TOTAL POINTS

95 / 100

QUESTION 1

Problem 1 - Page 1 20 pts

1.1 Part (a) 6 / 6

✓ + 1 pts **Correct definition of curl.**

✓ + 3 pts **Correct computation of curl.**

✓ + 1 pts **F is defined on R^3 (a simply connected domain).**

✓ + 1 pts **Conclusion that F is conservative.**

- 1 pts Notation error.

+ 2 pts Partial credit: Some of curl computation correct.

1.2 Part (b) 4 / 4

✓ + 2 pts **Correct function f.**

✓ + 2 pts **Some justification (either partial integrals or computation of the gradient).**

+ 1 pts Partial credit: part of f correct.

+ 0 pts Completely incorrect or blank.

1.3 Part (c) 5 / 5

✓ + 2 pts **Reference to Fundamental Theorem of Line Integrals.**

✓ + 2 pts **Correct formula involving endpoints.**

+ 1 pts Partial credit: Correct formula involving endpoints but with endpoints reversed.

✓ + 1 pts **Correct final answer.**

+ 0 pts Blank or completely incorrect.

1.4 Part (d) 5 / 5

✓ + 1 pts **div(curl G) = 0**

✓ + 1 pts **Correctly calculate div(F).**

✓ + 1 pts **Observe div(F) is not zero.**

✓ + 2 pts **Conclude no such G exists based on correct computation.**

+ 0 pts Blank or completely incorrect.

QUESTION 2

2 Problem 2 - Page 2 10 / 15

+ 15 pts All Correct

+ 2 pts (a) Correct dF1/dy

+ 2 pts (a) Correct dF2/dx

✓ + 2 pts **(a) (partial) One computational error in computing dF1/dy and dF2/dx, but otherwise correct**

✓ + 3 pts **(b) F not defined when x=y**

✓ + 2 pts **(b) Correctly stated the two domains (graphically or algebraically)**

+ 3 pts (b) Two domains are simply connected

✓ + 3 pts **(b) F is conservative on each domain (with proper reasoning)**

+ 0 pts Blank or no correct work

The term "simply connected" applies to sets, not to components of vector fields.

QUESTION 3

3 Problem 3 - Page 2 15 / 15

✓ + 15 pts **All Correct**

+ 2 pts (Partial credit) Tried to parametrize line

+ 4 pts Correctly parametrized line

+ 2 pts Computed r'(t) correctly

+ 3 pts Correct form for integrand

+ 2 pts Correct bounds for integral

+ 2 pts Evaluated F(r(t)) correctly

+ 1 pts Evaluated dot product correctly

+ 1 pts Evaluated integral correctly

+ 5 pts Computed that curl(F)=0 / showed that F was conservative.

+ 5 pts Computed potential function correctly

+ 5 pts Applied Fundamental Theorem of Line Integrals correctly

+ 0 pts Blank or no correct work

QUESTION 4

4 Problem 4 - Page 3 **25 / 25**

✓ + **25 pts** **Correct**

+ **2 pts** Closing Curve
+ **3 pts** Parameterizing Closing Line
+ **3 pts** Mentioning Green's Theorem
+ **4 pts** Using Green's Theorem Correctly
+ **5 pts** Correct Double Integral Setup
+ **1 pts** Correct Double Integral Computation (Given Reasonable Setup)
+ **5 pts** Correct Line Integral Setup (Given Reasonable Parameterization)
+ **1 pts** Correct Line Integral Computation (Given Reasonable Setup)
+ **1 pts** Correct Answer
+ **4 pts** Not Using Green's Theorem, but Setting Up Three Line Integrals
+ **3 pts** Almost Correct Double Integral Setup
+ **3 pts** Almost Correct Line Integral Setup
+ **1 pts** Almost Parameterizing the Curve Correctly
- **3 pts** Mixing up Partial Derivative Order in Green's Theorem
- **2 pts** Adding Line Integral Rather Than Subtracting
- **1 pts** Small Mistake In Calculation That Doesn't Affect Answer
+ **0 pts** Nothing Relating to the Problem There

QUESTION 5

5 Problem 5 - Page 4 **25 / 25**

✓ + **25 pts** **Correct Solution**

+ **2 pts** Formula: Wrote formula for surface integral in terms of parametrization (i.e. equation from Theorem 1 of the vector surface integral section)
+ **5 pts** Parametrization: Provided a valid parametrization of the surface
+ **2 pts** Normal Vector (Method 1): Calculated first tangent vector
+ **2 pts** Normal Vector (Method 1): Calculated second tangent vector
+ **1 pts** Normal Vector (Method 1): Computed cross product of tangent vectors to obtain normal vector
+ **5 pts** Normal Vector (Method 2, Full Credit): Correctly used the formula for normal vector of a surface given by a graph
+ **2 pts** Normal Vector (Method 2, Partial Credit): Used formula for normal vector of a graph but made mistakes
+ **2 pts** Orientation: Either had the normal vector field oriented correctly, OR identified that the normal vector field was incorrectly oriented and flipped sign of final answer
+ **3 pts** Integrand: Evaluated F dot N using a valid parametrization
+ **2 pts** Bounds of Integration: Correct bounds for 1st variable (not awarded if there are major errors in setup)
+ **2 pts** Bounds of Integration: Correct bounds for 2nd variable (not awarded if there are major errors in setup)
+ **2 pts** Computation: Evaluated 1st iterated integral
+ **2 pts** Computation: Evaluated 2nd iterated integral & obtained correct answer
+ **0 pts** No Credit

1. (20 points) Let $\mathbf{F}(x,y,z) = \langle 2xy^2z, 2x^2yz, x^2y^2 + 2z\rangle$ and let C be the line segment from $(1,1,3)$ to $(1,1,-3)$.

(a) Show that the vector field $\mathbf{F}$ is conservative using curl.

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2z & 2x^2yz & x^2y^2+2z \end{vmatrix}$$

$$= \hat{\imath}(2x^2y - 2x^2y) - \hat{\jmath}(2xy^2 - 2xy^2) + \hat{k}(4xyz - 4xyz)$$

$$\text{curl } \vec{F} = \langle 0,0,0\rangle$$

Since
- curl $\vec{F} = \vec{0}$
- $\vec{F}$ is defined on $\mathbb{R}^3$, a simply connected domain
- F_1, F_2, and F_3 have continuous first order partials, as shown in the calculation. (and F_1, F_2, F_3 are polynomial)

$\vec{F}$ is conservative

(b) Find a function f such that $\mathbf{F} = \nabla f$.

$f(x,y,z) = x^2y^2z + z^2 + C$ for constant number C

verify: $\nabla f = \langle 2xy^2z + 0, 2x^2yz + 0, x^2y^2 + 2z\rangle = \vec{F}$ ✓

(c) Use part (b) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$\int_C \vec{F}\cdot d\vec{r} = f(1,1,-3) - f(1,1,3)$ by FTC for line integrals ($\vec{F} = \vec{\nabla} f$)

$= (1)^2(1)^2(-3) + (-3)^2 - \left((1)^2(1)^2(3) + (3)^2\right)$

$= -3 + 9 - (3 + 9)$

$\int_C \vec{F}\cdot d\vec{r} = -6$

(d) Is there a vector field $\mathbf{G}$ defined on $\mathbb{R}^3$ such that $\text{curl}\,\mathbf{G} = \mathbf{F}$?

$$\text{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{div}(F) = 2y^2z + 2x^2z + 2 \neq 0$$

but, div(curl G) must equal 0 if G exists, (as proved in class)

and div(curl G) = div(F), but div(F) $\neq 0$.

This is a contradiction, so G cannot exist. No

/20

2. (15 points) Consider the vector field

$$\mathbf{F}(x,y) = \langle F_1, F_2\rangle = \left\langle \frac{2}{x-y}, \frac{2}{y-x} \right\rangle.$$

(a) Show that $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$.

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2}{x-y}\right) = (-1)\frac{-1}{(x-y)^2} = \frac{1}{(x-y)^2}$$

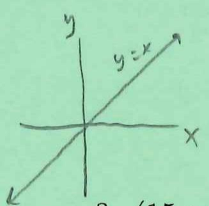
$\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ are equal

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2}{y-x}\right) = (-1)\frac{-1}{(y-x)^2} = \frac{1}{(y-x)^2} = \frac{1}{(-(x-y))^2} = \frac{1}{(x-y)^2}$$

(b) Show that $\mathbf{F}$ is defined on two distinct connected domains in the plane. On each of these domains, is $\mathbf{F}$ conservative? *Hint:* Are these domains simply connected?

F_1 undefined for $x-y=0 \Rightarrow x=y$

F_2 undefined for $y-x=0 \Rightarrow x=y$, because of division by zero.



F_1 and F_2 are continuous and simply connected everywhere but the line $y=x$. F_1 & F_2 have continuous 1st partials everywhere but $y=x$.

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\right)(\hat{i}) = \vec{0}.$$

because of the facts above, Yes $\vec{F}$ is conservative on the 2 distinct domains where $y \neq x$.

3. (15 points) Find the work done by the force field $\mathbf{F}(x,y,z) = \langle x^2, y^2, z^3\rangle$ in moving a particle along the line segment from $(0,0,0)$ to $(1,2,2)$.

$\vec{F} = \nabla f$ for $f(x,y,z) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{4}z^4 + C$. verify: $\nabla f = \langle \frac{3}{3}x^2, \frac{3}{3}y^2, \frac{4}{4}z^3 \rangle$ ✓.

work done by $\vec{F}$ = $\int_C \vec{F}\cdot d\vec{r} = f(1,2,2) - f(0,0,0)$ by FTC for line integrals. $(\vec{F} = \nabla f)$

line segment

$$= \tfrac{1}{3}(1)^3 + \tfrac{1}{3}(2)^3 + \tfrac{1}{4}(2)^4 - \left(\tfrac{1}{3}(0)^3 + \tfrac{1}{3}(0)^3 + \tfrac{1}{4}(0)^4\right)$$

$$= \tfrac{1}{3} + \tfrac{1}{3}\cdot 8 + \tfrac{1}{4}\cdot 16 - 0$$

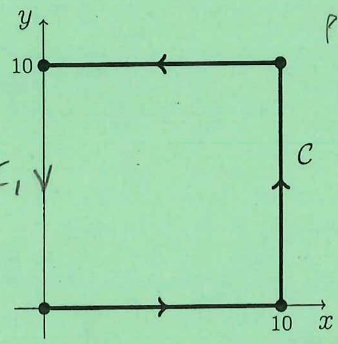
$$= \tfrac{1}{3} + \tfrac{8}{3} + 4$$

$$= 3 + 4$$

$$= 7$$

/30

4. (25 points) Let C be the curve given by the line segments from $(0,0)$ to $(10,0)$ to $(10,10)$ to $(0,10)$ as pictured below. Evaluate $\int_C (e^{x^2} + 3y)\,dx + (5x + 2y)\,dy$. *Hint:* Complete C to form a closed curve and use Green's Theorem.



parametrize C_1: $\vec{r}_1(t) = \langle 0, 10-t \rangle$ $\quad 0 \le t \le 10$

$\vec{r}_1{}'(t) = \langle 0, -1 \rangle$

path integral form: $\underbrace{\int_C F_1\,dx + F_2\,dy}$

$\vec{F} = \langle e^{x^2} + 3y, 5x + 2y \rangle$

$\frac{\partial F_2}{\partial x} = 5 \qquad \frac{\partial F_2}{\partial y} = 2$

$\frac{\partial F_1}{\partial y} = 3 \qquad \frac{\partial F_1}{\partial x} = 2xe^{x^2}$

Apply Green's theorem;

$C \cup C_1$ is
- positively oriented ✓
- piecewise smooth ✓
- simple ✓
- closed ✓

D is a simply connected domain enclosed by $C \cup C_1$ ✓

$\vec{F}$ has continuous 1st order partials ✓

so $\oint_{C \cup C_1} \vec{F} \cdot d\vec{r} \overset{\text{green's}}{=} \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$

$\int_C \vec{F} \cdot d\vec{r} + \int_{C_1} \vec{F} \cdot d\vec{r} = \iint_D 5 - 3\,dA$

$\int_C \vec{F} \cdot d\vec{r} + \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{10} \int_0^{10} 2\,dx\,dy$

$= \int_0^{10} [2x]_0^{10}\,dy$

$= \int_0^{10} 20 - 0\,dy$

$= [20y]_0^{10}$

$\int_C \vec{F} \cdot d\vec{r} + \int_{C_1} \vec{F} \cdot d\vec{r} = 200 - 0$

$\int_C \vec{F} \cdot d\vec{r} = 200 - \int_{C_1} \vec{F} \cdot d\vec{r}$

$\int_C \vec{F} \cdot d\vec{r} = 200 - \int_0^{10} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$= 200 - \int_0^{10} \langle e^{0^2} + 3(10-t), 5(0) + 2(10-t) \rangle \cdot \langle 0, -1 \rangle$

$= 200 - \int_0^{10} 0 + -1(0 + 20 - 2t)\,dt$

$= 200 - \int_0^{10} 0 - 20 + 2t\,dt$

$= 200 - \left[-20t + \frac{2}{2}t^2\right]_0^{10}$

$= 200 - \left(-20(10) + (10)^2 - 0\right)$

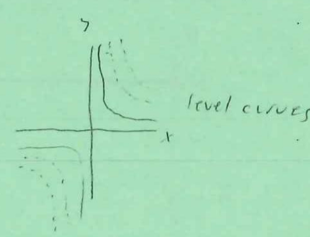
$= 200 + 200 - 100$

$\int_C \vec{F} \cdot d\vec{r} = 300$

/25

5. (25 points) Consider the vector field $\mathbf{F}(x,y,z) = \langle x^2, y^2, 4z\rangle$ and the surface S given by $z = xy$ for $0 \le x \le 1$ and $0 \le y \le 1$. Suppose S is oriented with upward normal. Find the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Parametrize S: $\vec{r}(u,v) = \langle u, v, uv \rangle$ D: $0 \le u \le 1$ $0 \le v \le 1$



$\vec{r}_u = \langle 1, 0, v \rangle$

$\vec{r}_v = \langle 0, 1, u \rangle$

$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \langle 0 - v, -u - 0, 1 - 0 \rangle$

$\vec{N}(u,v) = \langle -v, -u, 1 \rangle$. the z component is positive, so yes this is an upward normal ✓.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\, du\, dv$$

$$= \iint_D \langle (u)^2, (v)^2, 4(uv) \rangle \cdot \langle -v, -u, 1 \rangle\, du\, dv$$

$$= \int_0^1 \int_0^1 -vu^2 - uv^2 + 4uv\, du\, dv$$

$$= \int_0^1 \left[-\frac{v}{3}u^3 - \frac{1}{2}u^2v^2 + \frac{4}{2}u^2 v \right]_0^1 dv$$

$$= \int_0^1 -\frac{v}{3}(1)^3 - \frac{1}{2}(1)^2 v^2 + \frac{4}{2}(1)^2 v - (0)\, dv$$

$$= \left[-\frac{1}{3}\frac{1}{2}v^2 - \frac{1}{2}\cdot\frac{1}{3}v^3 + \frac{2}{2}v^2 \right]_0^1$$

$$= \frac{-1}{3 \cdot 2}(1)^2 - \frac{1}{2}\cdot\frac{1}{3}(1)^3 + \frac{2}{2}(1)^2 - (0)$$

$$= \frac{-1}{6} - \frac{1}{6} + 1$$

$$= \frac{-1}{3} + 1$$

$$= 2/3$$

/25