QUESTION 1

1 Problem 1 - Page 1 15 / 15

- ✓ + 4 pts **Switched the order of integration**
- ✓ + 1 pts **Correct lower bound for x**
- ✓ + 1 pts **Correct upper bound for x**
- ✓ + 1 pts **Correct lower bound for y**
- ✓ + 1 pts **Correct upper bound for y**
- ✓ + 2 pts **Justification for bounds (either graphical or algebraic)**
- ✓ + 2 pts **Correctly computed inner integral**
- ✓ + 2 pts **Correctly computed outer integral**
- ✓ + 1 pts **Correct final answer**
- + **0 pts** Blank or completely incorrect

QUESTION 2

2 Problem 2 - Page 1 10 / 10

- ✓ + 2 pts **Switched to polar coordinates.**
- ✓ + 1 pts **Correct bounds for theta**
- ✓ + 1 pts **Correct bounds for r**
- ✓ + 2 pts **dy dx = r dr dtheta (or r dtheta dr)**
- ✓ + 2 pts **Correctly rewrote integrand**
- ✓ + 1 pts **Correctly computed dr integral**
- ✓ + 1 pts **Correctly computed dtheta integral**
- + **0 pts** Blank or entirely incorrect

QUESTION 3

3 Problem 3 - Page 2 10 / 10

- + **1 pts** Demonstrating Understanding of Definition of Density Function (other than setting integral to 1)
- + **3 pts** Setting Integral Equal to 1
- + **2 pts** Correct Bounds on Integral
- + **2 pts** Correct Integral Computation for Reasonable Bounds
- + **2 pts** Correct C
- ✓ + 10 pts **Completely Correct**
- + **0 pts** Blank or Insufficient Work
- + **3 pts** Incorrect C as a Result of Small Computational Mistake for Completely Correct Integral
- + **1 pts** Integrating the function without anything above

QUESTION 4

4 Problem 4 - Page 2 15 / 15

- + **3 pts** Writing the Volume in Terms of an Integral of 1
- + **2 pts** Finding Projection onto x-z Plane
- + **4 pts** Correct Integral Bounds
- + **3 pts** Conversion to Cylindrical Coordinates
- + **2 pts** Correct Computation Given Reasonable Bounds
- + **1 pts** Correct Answer
- ✓ + 15 pts **Completely Correct Solution**
- + **0 pts** Blank or Insufficient Work
- + **2 pts** Correct Picture (given incorrect solution)
- + **1 pts** General Awareness of Shape we're Finding the Volume of
- + **2 pts** Almost Correct Integral Bounds

QUESTION 5

Problem 5 - Page 3 25 pts

5.1 Rectangular Coordinates 7 / 7

- ✓ + 1 pts **Correct lower bound for x integral**
- ✓ + 1 pts **Correct upper bound for x integral**
- ✓ + 1 pts **Correct lower bound for y integral**
- ✓ + 1 pts **Correct upper bound for y integral**
- ✓ + 1 pts **Correct lower bound for z integral**
- ✓ + 1 pts **Correct upper bound for z integral**
- ✓ + 1 pts **Correct integrand and order of integration**
- + **0 pts** Blank or completely incorrect

+ **1 pts** Partial credit: assumed projection was a disk with radius 2 but gave correct bounds for both x and y for that projection.

5.2 Cylindrical Coordinates 9 / 9

✓ + **1 pts** **Correct lower bound for theta**

✓ + **1 pts** **Correct upper bound for theta**

✓ + **1 pts** **Correct lower bound for r**

✓ + **1 pts** **Correct upper bound for r**

✓ + **1 pts** **Correct lower bound for z**

✓ + **1 pts** **Correct upper bound for z**

✓ + **1 pts** **Correct integrand substitution**

✓ + **1 pts** **Jacobian**

✓ + **1 pts** **Correct order of integration**

+ **0 pts** Blank or incorrect

5.3 Spherical Coordinates 9 / 9

✓ + **1 pts** **Correct lower bound for phi**

✓ + **1 pts** **Correct upper bound for phi**

✓ + **1 pts** **Correct lower bound for theta**

✓ + **1 pts** **Correct upper bound for theta**

✓ + **1 pts** **Correct lower bound for rho**

✓ + **1 pts** **Correct upper bound for rho**

✓ + **1 pts** **Correct integrand substitution**

✓ + **1 pts** **Jacobian**

✓ + **1 pts** **Correct order of integration**

+ **0 pts** Blank or incorrect

QUESTION 6

6 Problem 6 - Page 4 10 / 10

✓ + **2 pts** **(Full credit) Correct drawing of at least 1 loop AND/OR deduction of bounds from properties of the sine function**

+ **1 pts** (Partial credit) Correct but vague drawing OR incorrect drawing OR insufficient description of region (e.g. not enough inequalities) OR mistake in description of region

✓ + **1 pts** **Identified area as the integral and/or wrote the integral of 1 over the region**

✓ + **1 pts** **Correct upper bound first iterated integral**

✓ + **1 pts** **Correct lower bound for first iterated integral**

✓ + **1 pts** **Correct upper bound for second iterated integral (for either the full loop, or half of the loop if you are doubling area at the end)**

✓ + **1 pts** **Correct lower bound for second iterated integral**

✓ + **1 pts** **Included the polar coordinates jacobian**

✓ + **2 pts** **(Full credit) Correct calculation of correct integrals and final answer**

+ **1 pts** (Partial credit) calculation of integrals and final answer: some mistakes in calculating correctly set up integral OR correct calculation of integral with small error in setup

+ **0 pts** Blank or completely incorrect or not enough work shown

QUESTION 7

7 Problem 7 - Page 4 15 / 15

✓ + **2 pts** **(Full credit) Correct change of variable (Circular to Elliptical)**

+ **1 pts** (Partial credit) Change of variable: minor error in circular to elliptical change of variable OR a change of variable which is impractical

✓ + **2 pts** **Correct description of new domain (Circular to Elliptical): At minimum, identify boundary as belonging to unit circle.**

✓ + **2 pts** **(Full credit) Correct jacobian calculation (Circular to Elliptical)**

+ **1 pts** (Partial credit) jacobian calculation: Error in calculation OR using reciprocal in the integral OR correctly calculating jacobian for impractical change of variable

✓ + **2 pts** **Change to polar coordinates (including jacobian)**

✓ + **1 pts** **Correct upper bound first iterated integral (not cartesian)**

✓ + **1 pts** **Correct lower bound for first iterated integral (not cartesian)**

✓ + **1 pts** **Correct upper bound for second iterated integral (not cartesian)**

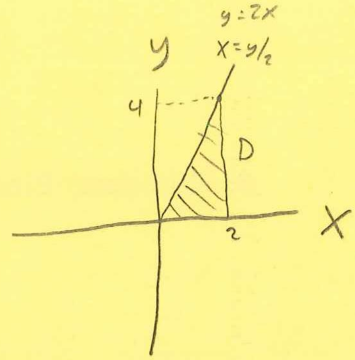
✓ + **1 pts** **Correct lower bound for second iterated integral (not cartesian)**

✓ + **1 pts** **Correct integrand in polar coordinates (aside from jacobian)**

✓ + **2 pts** **(Full credit) Correct calculation of correct integrals and final answer**

+ **1 pts** (Partial credit) calculation of integrals and final answer: some mistakes in calculating correctly set up integral OR correct calculation of integral with small error in setup

+ **0 pts** Blank or completely incorrect or not enough work shown



1. (15 points) Evaluate the iterated integral

$$\int_0^4\int_{y/2}^2 \cos\left(\frac{\pi}{24}x^2\right)\,dx\,dy.$$

Since $\cos(\pi/24\, x^2)$ is continuous, we can evaluate the iterated integral in any order by Fubini's theorem.

$$= \int_0^2\int_0^{2x} \cos(\pi/24\, x^2)\,dy\,dx$$

$$= \int_0^2 \left[y\cos\left(\tfrac{\pi}{24}x^2\right)\right]_{y=0}^{y=2x} dx$$

$$= \int_0^2 2x\cos\left(\tfrac{\pi}{24}x^2\right) - 0\cos\left(\tfrac{\pi}{24}x^2\right)dx$$

let $u = \pi/24\, x^2$

$du = \frac{\pi}{24}\cdot 2x\,dx$

$\frac{24}{\pi}du = 2x\,dx$

$$= \int_{x=0}^{x=2} \cos(u)\,\frac{24}{\pi}\,du$$

$$= \left[\sin(u)\,\frac{24}{\pi}\right]_{x=0}^{x=2}$$

$$= \sin\left(\pi/24\, x^2\right)\frac{24}{\pi}\Big]_{x=0}^{x=2}$$

$$= \sin\left(\tfrac{\pi}{24}\cdot 2^2\right)\frac{24}{\pi} - \sin\left(\pi/24\cdot 0^2\right)\frac{24}{\pi} = \sin\left(\tfrac{\pi}{6}\right)\frac{24}{\pi} - 0 = \frac{12}{\pi}$$

2. (10 points) Evaluate the iterated integral.

$$\int_{-3}^3\int_0^{\sqrt{9-x^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dy\,dx$$

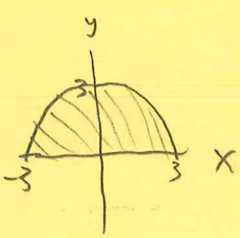
$y = \sqrt{9-x^2}$ | $y = 0$

$x^2+y^2 = 9$ | $r\sin\theta = 0$

$r^2 = 9$ | $r = 0$

$r = 3$

Switch to polar coordinates



$$= \int_0^\pi\int_0^3 \frac{1}{\sqrt{1+r^2}}\cdot r\,dr\,d\theta$$

$$= \int_0^\pi \int_{r=0}^{r=3} \frac{1}{\sqrt{u}}\cdot\frac{1}{2}\,du\,d\theta$$

let $u = 1+r^2$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

$u^{-1/2}$

$$= \int_0^\pi \left[\frac{1}{1/2}u^{1/2}\cdot\frac{1}{2}\right]_{r=0}^{r=3} d\theta$$

$$= \int_0^\pi 2\cdot\frac{1}{2}(1+3^2)^{1/2} - 2\cdot\tfrac{1}{2}(1+0^2)^{1/2}\,d\theta$$

$$= \int_0^\pi \sqrt{10} - \sqrt{1}\,d\theta$$

$$= \left[(\sqrt{10}-1)\theta\right]_{\theta=0}^{\theta=\pi}$$

$$= \pi(\sqrt{10}-1)$$

/25

$0 \le y \le 1$

$y \le x$

3. (10 points) Find a constant C such that $y \le x \le 1$

$$p(x,y) = \begin{cases} Cx^3y & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint probability density function.

To be a joint probability density function: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx = 1$

since $p(x,y)=0$ everywhere but on D: $\iint_D Cx^3y\,dy\,dx = 1$

$\int_0^1\int_0^x Cx^3y\,dy\,dx = 1$

$C\int_0^1 \left[x^3 \cdot \frac{1}{2}y^2\right]_{y=0}^{y=x} dx = 1$

$\frac{C}{2}\int_0^1 x^3(x)^2 - x^3(0)^2\,dx = 1$

$\frac{C}{2}\int_0^1 x^5\,dx = 1$

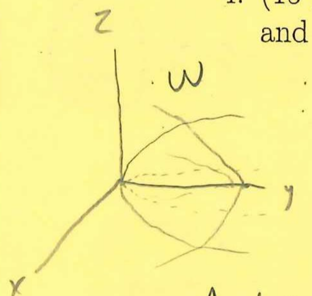
$\frac{C}{2}\left[\frac{1}{6}x^6\right]_0^1 = 1$

$\frac{C}{12}(1^6 - 0^6) = 1$

$\frac{C}{12} = 1$

$C = 12$

4. (15 points) Use a triple integral to find the volume of the solid enclosed by $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.



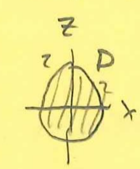
intersection of surfaces: $x^2+z^2 = 8-x^2-z^2$

$2x^2+2z^2 = 8$

$x^2+z^2 = 4$

projection D onto xz plane:

$D: x^2+z^2 \le 4$



Analyze as y-simple

$\text{Vol}(W) = \iiint_W 1\,dV = \iint_D \left(\int_{x^2+z^2}^{8-x^2-z^2} 1\,dy\right) dA$

switch to cylindrical coords

$x^2+z^2 = r^2$

$y = y$

$= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} 1 \cdot r\,dy\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 [ry]_{y=r^2}^{y=8-r^2}\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 r(8-r^2) - r(r^2)\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 8r - r^3 - r^3\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 8r - 2r^3\,dr\,d\theta$

$= \int_0^{2\pi}\left[8 \cdot \frac{1}{2}r^2 - 2 \cdot \frac{1}{4}r^4\right]_{r=0}^{r=2} d\theta$

$= \int_0^{2\pi}\left(\frac{8}{2}(2)^2 - \frac{2}{4}(2)^4 - \left(\frac{8}{2}(0)^2 - \frac{2}{4}(0)^4\right)\right) d\theta$

$= \int_0^{2\pi} 4 \cdot 4 - \frac{1}{2} \cdot 16\,d\theta$

$= \int_0^{2\pi} 16 - 8\,d\theta$

$= \int_0^{2\pi} 8\,d\theta$

$= [8\theta]_0^{2\pi}$

$= 2\pi \cdot 8 - 8 \cdot 0$

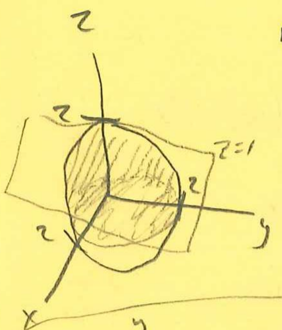
$= 16\pi$

/25

5. (25 points) Let $\mathcal{W}$ be the solid inside the sphere $x^2 + y^2 + z^2 = 4$ for $z \geq 1$. Set up but **DO NOT EVALUATE** a triple integral in each of the following coordinate systems that computes the mass of the solid $\mathcal{W}$, assuming it has density function $\delta(x, y, z) = 5xy$.

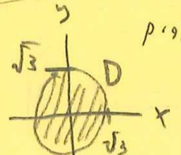
1. Rectangular coordinates

$$\text{mass}(w) = \iiint_w \delta(x,y,z)\,dV = \iint_D \left(\int_1^{\sqrt{4-x^2-y^2}} 5xy\,dz\right) dA$$



$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} 5xy\,dz\,dy\,dx$$

projection D



$x^2+y^2+1^2=4$

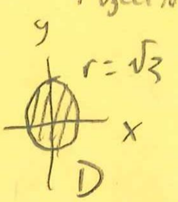
$x^2+y^2=3$

2. Cylindrical coordinates

$$\text{mass}(w) = \iiint_w \delta(x,y,z)\,dv = \iint_D \left(\int_1^{\sqrt{4-x^2-y^2}} 5xy\,dz\right) dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 5(r\cos\theta)(r\sin\theta)\cdot r\,dz\,dr\,d\theta$$

Projection



$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 5r^3 \sin\theta\cos\theta\,dz\,dr\,d\theta$$

3. Spherical coordinates

spherical

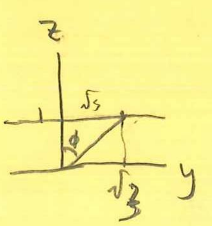
$x^2+y^2+z^2 \leq 4 \to \rho^2 \leq 4 \to \rho \leq 2$

$z \geq 1 \to \rho\cos\phi \geq 1 \to \rho \geq \frac{1}{\cos\phi}$

$$\text{mass}(w) = \iiint_w 1\,dV = \int_0^{\pi/3} \int_0^{2\pi} \int_{\frac{1}{\cos\phi}}^{2} 5xy \cdot \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/3} \int_0^{2\pi} \int_{1/\cos\phi}^{2} 5(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

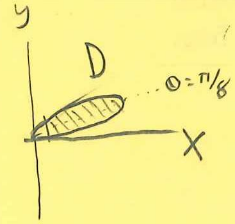
$\phi = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \pi/3$



$$= \int_0^{\pi/3} \int_0^{2\pi} \int_{\frac{1}{\cos\phi}}^{2} 5\rho^4 \sin^3\phi\,\sin\theta\cos\theta\,d\rho\,d\theta\,d\phi$$

/25

6. (10 points) Use a double integral to find the area inside one loop of the polar rose $r = 5\sin(4\theta)$. *Hint:* You may use the double angle formula $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$.



r is maximized at $\sin(4\theta) = \sin(\pi/2)$
$4\theta = \pi/2$
$\theta = \pi/8$

r is 0 again at $\sin(4\theta) = \sin(\pi)$
$4\theta = \pi$
$\theta = \pi/4$

$\text{Area}(D) = \iint_D 1\, dA$

$= \int_0^{\pi/4} \int_0^{5\sin(4\theta)} 1 \cdot r\, dr\, d\theta$

$= \int_0^{\pi/4} \left[\frac{1}{2} r^2\right]_0^{5\sin(4\theta)} d\theta$

$= \int_0^{\pi/4} \frac{1}{2}(5\sin(4\theta))^2 - \frac{1}{2}(0)^2\, d\theta$

$= \int_0^{\pi/4} \frac{1}{2} \cdot 25 \sin^2(4\theta)\, d\theta$

$= \frac{25}{2} \int_0^{\pi/4} \frac{1}{2} - \frac{\cos(2(4\theta))}{2}\, d\theta$

$= \frac{25}{2} \int_0^{\pi/4} \frac{1}{2}\, d\theta - \frac{25}{2} \int_0^{\pi/4} \frac{1}{2} \cos(8\theta)\, d\theta$

$= \frac{25}{4} \int_0^{\pi/4} 1\, d\theta - \frac{25}{4} \int_0^{\pi/4} \cos(8\theta)\, d\theta$

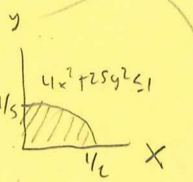
$= \frac{25}{4} [\theta]_0^{\pi/4} - \frac{25}{4} \left[\frac{1}{8}\sin(8\theta)\right]_0^{\pi/4}$

$= \frac{25}{4}(\pi/4 - 0) - \frac{25}{4}\left(\frac{1}{8}\sin(8 \cdot \frac{\pi}{4}) - \frac{1}{8}\sin(8 \cdot 0)\right)$

$= \frac{25}{4} \cdot \frac{\pi}{4} - \left(\frac{25}{4} \cdot \frac{1}{8} \underbrace{\sin(2\pi)}_{=0} - 0\right)$

$= \frac{25}{16}\pi$

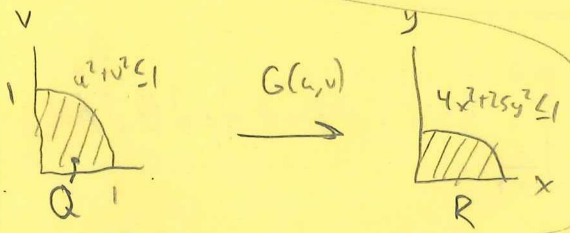
7. (15 points) Use a change of variables to evaluate $\iint_{\mathcal{R}} \cos\left(4x^2 + 25y^2\right)\, dA$ where $\mathcal{R}$ is the region in the first quadrant of the xy-plane bounded by the ellipse $4x^2 + 25y^2 = 1$.



$4x^2 + 25y^2 = 1$
$(2x)^2 + (5y)^2 = 1$
$u^2 + v^2 = 1$

$\rightarrow$ let $u = 2x$, $v = 5y$ $\longrightarrow$ $x = \frac{1}{2}u$, $y = \frac{1}{5}v$. $\rightarrow G(u,v) = \left(\frac{1}{2}u, \frac{1}{5}v\right)$

$J(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{vmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{5}\right) - (0)(0) = \frac{1}{10}$



$\iint_R \cos(4x^2 + 25y^2)\, dA$

$= \iint_Q \cos(u^2 + v^2)\, |J|\, du\, dv$ switch to polar in u and v $u^2 + v^2 = r^2$

$= \int_0^{\pi/2} \int_0^1 \cos(r^2)\left(\frac{1}{10}\right) \cdot r\, dr\, d\theta$

$\frac{1}{10} \int_0^{\pi/2} \left[\frac{1}{2}\sin(r^2)\right]_0^1 d\theta$

$\frac{1}{10} \int_0^{\pi/2} \frac{1}{2}\sin(1^2) - \frac{1}{2}\sin(0)^2\, d\theta$

$= \frac{1}{10} \int_0^{\pi/2} \frac{1}{2}\sin(1)\, d\theta$

$= \frac{1}{10} \frac{1}{2} \sin(1)(\pi/2 - 0)$

$= \frac{\pi}{40}\sin(1)$

/25