Math 32B - Fall 2019
Practice Exam 1

Full Name: Solutions

UID: ____________________

Circle the name of your TA and the day of your discussion:

Steven Gagniere   Jason Snyder   Ryan Wilkinson

Tuesday   Thursday

Instructions:

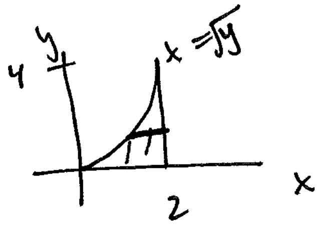
- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- Simplify your answers as much as possible.
- Include units with your answer where applicable.
- Calculators are not allowed but you may have a 3×5 inch notecard.

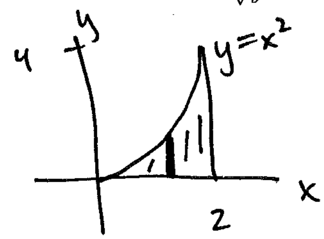
Page	Points	Score
1	20	
2	20	
3	15	
4	20	
5	25	
Total:	100	

1. (10 points) Evaluate the iterated integral.

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy$$

no easy antiderivative





both vertically simple and horizontally simple

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy = \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx = \int_0^2 \Big[y\sqrt{x^3+1}\Big]_0^{x^2} dx$$

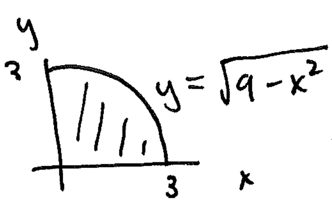
$$= \int_0^2 x^2\sqrt{x^3+1}\,dx = \Big[\tfrac{1}{2}\cdot\tfrac{2}{3}(x^3+1)^{3/2}\Big]_0^2 = \tfrac{2}{9}\Big[(x^3+1)^{3/2}\Big]_0^2$$

$$= \tfrac{2}{9}\left(9^{3/2}-1^{3/2}\right) = \tfrac{2}{9}(27-1) = \frac{2\cdot 26}{9} = \boxed{\frac{52}{9}}$$

2. (10 points) Evaluate the iterated integral.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} e^{x^2+y^2}\,dy\,dx$$

no elementary antiderivative



$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

polar

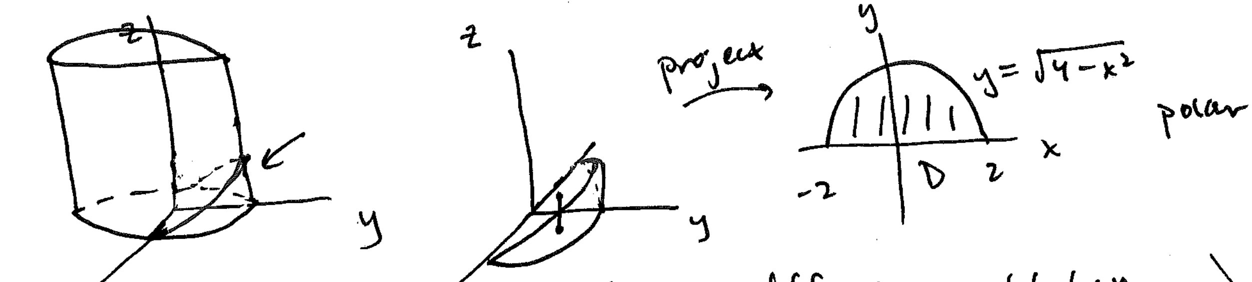
$$\int_0^3 \int_0^{\sqrt{9-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^3 e^{r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^3 re^{r^2}\,dr$$

$$= \Big[\theta\Big]_0^{\frac{\pi}{2}} \cdot \Big[\tfrac{1}{2}e^{r^2}\Big]_0^3$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\left(e^9-e^0\right) = \boxed{\frac{\pi}{4}\left(e^9-1\right)}$$

3. (10 points) Find the volume of the solid enclosed by $z = 0$, $y = z$, and $x^2 + y^2 = 4$.



z-simple (also y-simple)

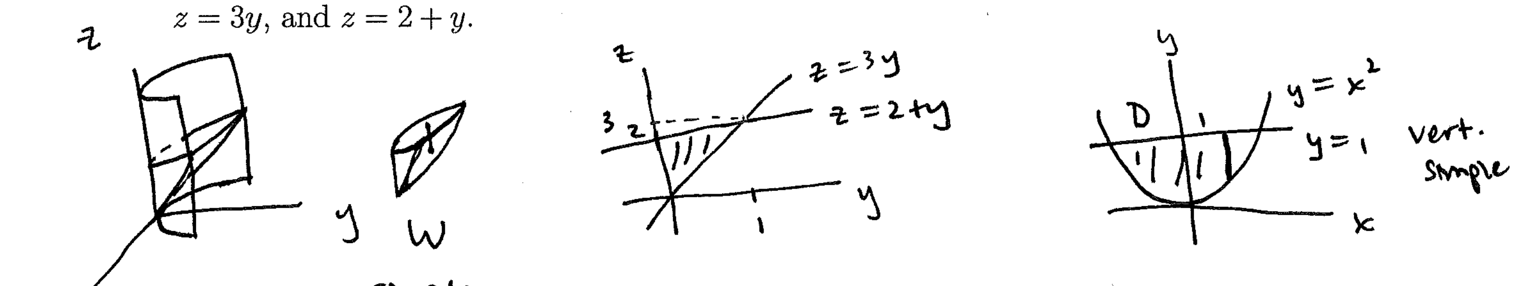
$$V = \iiint_W 1\, dV = \iint_D \left(\int_0^y 1\, dz \right) dA \quad \text{so}$$

$$V = \iint_D y\, dA = \int_0^{\pi} \int_0^2 r\sin\theta \cdot r\, dr\, d\theta = \int_0^{\pi} \sin\theta\, d\theta \cdot \int_0^2 r^2\, dr$$

$$= \left[-\cos\theta\right]_0^{\pi} \cdot \left[\tfrac{1}{3} r^3\right]_0^2 = \left(-\overset{-1}{\cos(\pi)} + \overset{1}{\cos(0)}\right)\left(\tfrac{1}{3}8 - 0\right)$$

$$= 2 \cdot \tfrac{1}{3} \cdot 8 = \boxed{\tfrac{16}{3}}$$

4. (10 points) Use a triple integral to find the volume of the solid enclosed by $y = x^2$, $z = 3y$, and $z = 2 + y$.



z-simple

$$z = 3y = 2 + y$$
$$2y = 2$$
$$y = 1$$

$$V = \iiint_W 1\, dV = \iint_D \left(\int_{3y}^{2+y} 1\, dz \right) dA = \int_{-1}^{1} \int_{x^2}^{1} \int_{3y}^{2+y} 1\, dz\, dy\, dx$$

$$= \int_{-1}^{1} \int_{x^2}^{1} \left[z\right]_{3y}^{2+y} dy\, dx = \int_{-1}^{1} \int_{x^2}^{1} 2 + y - 3y\, dy\, dx$$

$$= \int_{-1}^{1} \int_{x^2}^{1} 2 - 2y\, dy\, dx = \int_{-1}^{1} \left[2y - y^2\right]_{x^2}^{1} dx = \int_{-1}^{1} (2-1) - (2x^2 - x^4)\, dx$$

$$= \int_{-1}^{1} 1 - 2x^2 + x^4\, dx = \left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_{-1}^{1} = \left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) - \left(-1 + \tfrac{2}{3} - \tfrac{1}{5}\right)$$

$$= 2 - \tfrac{4}{3} + \tfrac{2}{5} = \frac{30 - 20 + 6}{15} = \boxed{\tfrac{16}{15}}$$

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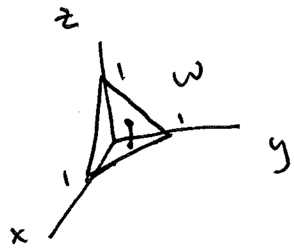
5. (15 points) Consider the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$ with density function $\delta(x,y,z)=12y$.

$z = 1-x-y$

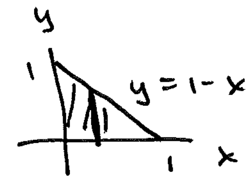
1. Find the mass of the tetrahedron.



z-simple

(also x-simple and y-simple)

if $z=0$, $x+y=1$ so $y=1-x$



$$m = \iiint_W \delta \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 12y \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \Big[12yz\Big]_0^{1-x-y} dy\, dx = \int_0^1 \int_0^{1-x} 12y(1-x-y)\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} 12y(1-x) - 12y^2 \, dy\, dx = \int_0^1 \Big[6y^2(1-x) - 4y^3\Big]_0^{1-x} dx$$

$$= \int_0^1 6(1-x)^3 - 4(1-x)^3 \, dx = \int_0^1 2(1-x)^3 dx$$

$$= \Big[-\tfrac{2}{4}(1-x)^4\Big]_0^1 = 0 + \tfrac{1}{2} = \boxed{\tfrac{1}{2}}$$

2. Set up but **DO NOT EVALUATE** the integrals used to find the center of mass of the tetrahedron.

center of mass is (x_{CM}, y_{CM}, z_{CM}) with

$$x_{CM} = \frac{1}{m} \iiint_W x\,\delta(x,y,z)\, dV = 2\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 12xy \, dz\, dy\, dx$$

$$y_{CM} = \frac{1}{m} \iiint_W y\,\delta(x,y,z)\, dV = 2\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 12y^2 \, dz\, dy\, dx$$

$$z_{CM} = \frac{1}{m} \iiint_W z\,\delta(x,y,z)\, dV = 2\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 12yz \, dz\, dy\, dx$$

6. (20 points) Evaluate the triple integral $\iiint_E x^2\,dV$ where E is the solid above $z = 0$ and inside $4x^2 + 9y^2 + z^2 = 36$.

ellipsoid

$2x = u,\ 3y = v,\ z = w$ $\quad G^{-1}$ $\quad \rightarrow u^2+v^2+w^2 = 36$ $\quad W$

$x = \frac{u}{2},\ y = \frac{v}{3},\ z = w$ $\quad G$ $\quad$ sphere $\rho = 6$ above $w = 0$

$$J(G) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{6} \qquad |J| = \frac{1}{6}$$

$$\text{so } \iiint_E x^2\,dV = \iiint_W \left(\frac{u}{2}\right)^2 \cdot \frac{1}{6}\,du\,dv\,dw = \frac{1}{24}\iiint_W u^2\,du\,dv\,dw$$

now spherical coordinates

$u = \rho\sin\phi\cos\theta,\quad v = \rho\sin\phi\sin\theta,\quad w = \rho\cos\phi$ $\qquad 0 \le \rho \le 6,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2}$

$$\iiint_E x^2\,dV = \frac{1}{24}\iiint_W u^2\,du\,dv\,dw = \frac{1}{24}\int_0^{\pi/2}\int_0^{2\pi}\int_0^6 \rho^2\sin^2\phi\cos^2\theta \cdot \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \frac{1}{24}\int_0^{\pi/2}\int_0^{2\pi}\int_0^6 \rho^4\sin^3\phi\cos^2\theta\,d\rho\,d\theta\,d\phi$$

$$= \frac{1}{24}\int_0^{\pi/2}\sin^3\phi\,d\phi \cdot \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^6 \rho^4\,d\rho$$

$$= \frac{1}{24}\int_0^{\pi/2}\sin\phi(1-\cos^2\phi)\,d\phi \cdot \int_0^{2\pi}\frac{1}{2}(1+\cos(2\theta))\,d\theta \int_0^6 \rho^4\,d\rho$$

$$= \frac{1}{24}\cdot\frac{1}{2}\int_0^{\pi/2}\sin\phi - \sin\phi\cos^2\phi\,d\phi \int_0^{2\pi} 1+\cos(2\theta)\,d\theta \int_0^6 \rho^4\,d\rho$$

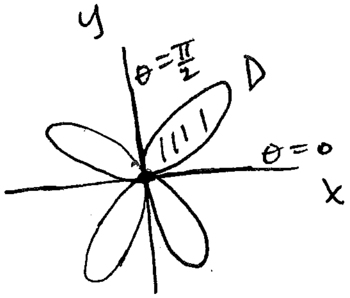
$$= \frac{1}{24}\cdot\frac{1}{2}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/2}\cdot\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{2\pi}\cdot\left[\frac{\rho^5}{5}\right]_0^6$$

$$= \frac{1}{24}\cdot\frac{1}{2}\left(0+0+1-\frac{1}{3}\right)(2\pi+0-0-0)\left(\frac{6^5}{5}\right)$$

$$= \frac{1}{\cancel{4}\cdot\cancel{6}}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\cancel{2}\pi\cdot\frac{6^{\cancel{5}\,3}}{5} = \frac{6^3\pi}{5} = \boxed{\frac{216\pi}{5}}$$

/20

7. (10 points) Find the area inside one petal of the polar rose $r = \sin(2\theta)$.



$0 \le r \le \sin(2\theta)$

$0 \le \theta \le \frac{\pi}{2}$

$$A = \iint_D 1\,dA = \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} 1 \cdot r\,dr\,d\theta$$

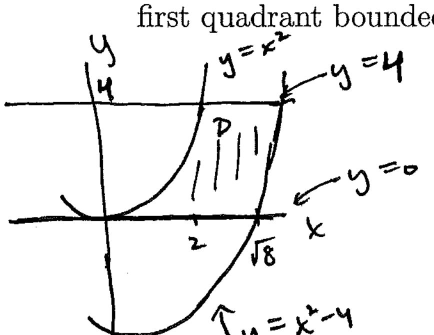
$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\sin(2\theta)} d\theta = \int_0^{\pi/2} \frac{\sin^2(2\theta)}{2}\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(\frac{1-\cos(4\theta)}{2}\right) d\theta = \frac{1}{4}\int_0^{\pi/2} 1-\cos(4\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2} = \frac{1}{4}\left(\pi/2 - 0 - 0 + 0\right)$$

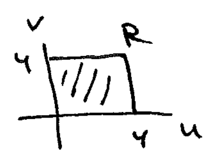
$$= \boxed{\frac{\pi}{8}}$$

8. (15 points) Use a <u>change of variables</u> to evaluate $\iint_{\mathcal{D}} x\,dA$ where $\mathcal{D}$ is the region in the first quadrant bounded by $y = 0$, $y = 4$, $y = x^2$, and $y = x^2 - 4$.



horizontally simple but required to use a change of variables

$0 \le y \le 4, \quad 0 \le x^2 - y \le 4$



So let $u = y, \ v = x^2 - y$ then $0 \le u \le 4, \ 0 \le v \le 4$

G^{-1}

$$J(G^{-1}) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2x & -1 \end{vmatrix} = -2x \quad \text{in region } D \ x \ge 0$$

$$J(G) = \frac{1}{J(G^{-1})} = -\frac{1}{2x}, \quad |J(G)| = \frac{1}{2x}.$$

(can solve $x = \sqrt{u+v}$ but not necessary)

$$\iint_D x\,dA = \iint_R x \cdot \frac{1}{2x}\,du\,dv = \iint_R \frac{1}{2}\,du\,dv = \frac{1}{2}\text{Area}(R) = \frac{1}{2}(4^2) = \boxed{8}$$

<u>check</u>: $\int_0^4 \int_{\sqrt{y}}^{\sqrt{y+4}} x\,dx\,dy = \int_0^4 \left[\frac{x^2}{2}\right]_{\sqrt{y}}^{\sqrt{y+4}} dy = \frac{1}{2}\int_0^4 y+4-y\,dy = \frac{1}{2}\int_0^4 4\,dy$
$= 8$

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