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Section	3	C

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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Spherical coordinates:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

This derivative might be useful:

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

Page:	1	2	3	4	5	Total
Points:	10	10	10	8	12	50
Score:						

You may use this page for scratch work.

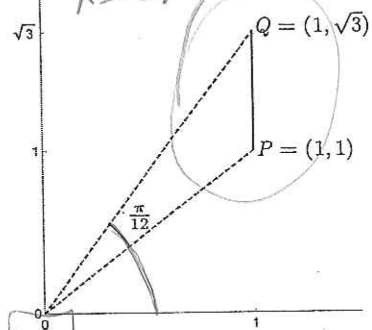
$\frac{1}{x^2+y^2}$ $\qquad$ $\frac{1}{1+x^2} = \arctan x$

1. (10 points) Let $\boldsymbol{F}$ denote the vortex field $\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Let C be the straight line segment from $P = (1,1)$ to $Q = (1,\sqrt{3})$ (see the picture). Show that $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{\pi}{12}$.

we can consider this conservative b/c (0,0) not in domain → (define domain as $x>0$, $y>0$.)

~~If we take~~



$\int F \cdot dr = f(1,\sqrt{3}) - f(1,1)$

$f = \tan^{-1}\left(\frac{y}{x}\right)$

$f(1,\sqrt{3}) - f(1,1) =$

$\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) - \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \boxed{\frac{\pi}{12}}$

$\frac{\pi}{4}$ $\quad$ $\pi/4$ $\quad$ $\pi/3$ $\quad$ $\sqrt{3}$ $\quad$ $\boxed{\frac{\pi}{12}}$

(b) Suppose that C is the ellipse parametrized by $\boldsymbol{r}(t) = \langle 5\cos(2t), 2\sin(2t)\rangle$ for $0 \le t \le \pi$. $\langle -10\sin(2t), 4\cos(2t)\rangle$

Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$. Box your answer.

$r(t) = \langle 5\cos(2t), 2\sin(2t)\rangle$

$r'(t) = \langle -10\sin(2t), 4\cos 2t\rangle$

~~$\|r'(t)\| = \sqrt{(-10\sin(2t))^2 + (4\cos 2t)^2}$~~

~~$= \sqrt{100\sin^2 2t + 16\cos^2 t}$~~

~~$= \sqrt{(100+16)}\,1$~~

~~$= \sqrt{116}$~~

$5^2\cos^2(2t) + 2^2\sin^2 2t$
$= 25 + 4 = 29$

$F(r(t)) = \left(\frac{-2\sin(2t)}{29}, \frac{5\cos(2t)}{29} \right)$

$\int_0^{\pi} \left\langle \frac{-2\sin(2t)}{29}, \frac{5\cos(2t)}{29} \right\rangle \cdot \langle -10\sin(2t), 4\cos(2t)\rangle \, dt$

$= \int_0^{\pi} \frac{20\sin^2 2t}{29} + \frac{20\cos^2 2t}{29} \, dt$

$\frac{20}{29} \int_0^{\pi} \sin^2(2t) + \cos^2(2t) \, dt$

$\frac{20}{29} \int_0^{\pi} dt$

$\frac{20}{29}\left(t \Big|_0^{\pi} \right) = \boxed{\frac{20}{29}\pi}$

(c) Is the vortex field $\boldsymbol{F}$ conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.

No, because the vector field is not defined at (0,0), but even just discluding that point gives us a hole in the domain, making the domain $\mathbb{R}^2/\{0,0\}$ not simply connected therefore F can't be conservative, even if f has a potential function defined on the domain $\mathbb{R}^2 \{0,0\}$.

2. (10 points) You do not need to simplify your answers.

(a) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where C is the straight line from $P = (1,2,3)$ to $Q = (4,5,6)$ and

$$\boldsymbol{F} = \left\langle \frac{2xy}{x^2+z}, \ln(x^2+z), \frac{y}{x^2+z} \right\rangle.$$

Box your answer

$r(t) = \langle t, t+1, t+2 \rangle$

If you define the domain as $z \geq 0$, $x > 0$, F is a conservative vector field on that domain and $P \to Q$ exists in that domain.

$\frac{F_1}{dz} = (x^2+z)^{-1} \cdot 2xy / (x^2+z)^2$

$\frac{F_3}{dx} = \frac{-2xy}{(x^2+z)^2}$

$\frac{y}{(x^2+z)}$

$\frac{1}{y(x^2+z)}$ $f = y(\ln(x^2+z))$

$f(4,5,6) - f(1,2,3)$

$\int \frac{1}{x} = \ln x$ anti deriv deriv

$\frac{F_1}{\partial y} = \frac{2x}{x^2+z}$ $\frac{F_2}{\partial x} = \frac{1}{x^2+z} \cdot 2x$

$5(\ln(16+6)) - 2\ln(1+3)$

$= \boxed{5\ln(22) - 2\ln(4)}$

(b) Suppose that $\boldsymbol{F} = \langle y, z, 0 \rangle$ and that S is the surface parametrized by

$$G(u,v) = (u^3 - v, u+v, v^2), \qquad 0 \leq u \leq 2, \quad 0 \leq v \leq 3.$$

with downward-pointing normal.

Fill in the limits and integrand of the integral below so that it equals $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

$T_u = \langle 3u, 1, 0 \rangle$

$T_v = \langle -1, 1, 2v \rangle$

$$\begin{matrix} i & j & k \\ 3u & 1 & 0 \\ -1 & 1 & 2v \end{matrix}$$

$F \cdot N = \langle y, z, 0 \rangle = \langle u+v, v^2, 0 \rangle$

$\cancel{\langle y, z, 0 \rangle} \cdot \langle$

$\langle u+v, v^2, 0 \rangle \cdot \langle -2v, +6uv, -3u-1 \rangle$

$-2uv - 2v^2, + 6uv^3$

$N = \langle 2v, -6uv, 3u+1 \rangle$ this is upward pointing

$N = \langle -2v, 6uv, -3u - 1 \rangle$ ← this is downward

$$\boxed{\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_0^3 \int_0^2 -2uv - 2v^2 + 6uv^3 \; du\, dv}$$

You do not need to show work on this page.

3. (10 points) (a) Give a parametrization $G : D \to S$, where S is the triangle in $\mathbb{R}^3$ with vertices $(2,0,0)$, $(0,1,0)$, $(0,0,1)$ in the plane $\frac{1}{2}x + y + z = 1$. Be sure to explicitly specify the domain D and call your parameters u and v.

$x=2 \quad y=1, \quad z=1 \qquad z = 1 - \frac{x}{2} + y$

$\frac{1}{2}\cancel{u+} \quad z = 1 - \frac{u}{2} + v$

$$\boxed{D = \{ 0 \le u \le 2, \quad 0 \le v \le 1 \}}$$

$$\boxed{G(u,v) = \left\langle u, \quad v, \quad 1 - \frac{u}{2} + v \right\rangle}$$

(b) Let S be the portion of the cylinder $x^2 + y^2 = 1$ between the xy-plane and the plane $x + z = 2$. We parametrize S by $G(u,v) = \langle \cos u, \sin u, v \rangle$, where (fill these in)

$$\boxed{0 \le u \le 2\pi, \quad 0 \le v \le 2-\cos u}$$

$x - y$

$z = 0$

$x + z = 2$

$\downarrow$

$x = \cos u$

$z = 2 - \cos u$

$\cancel{\cos^2 u}$

$\cancel{-\sin u, -\cos u}$

$T_u = -\sin u, \cos u \ 0$

$T_v = 0, \ 0, \ 1$

$(\cos u, \sin u, 0)$

$\int_0^{2\pi}\int_0^{2} x \, ds$

$\int_0^{2\pi}\int_0^{2} \cos^2 u + \cos u \sin u$

$\int_0^{2\pi} \cos^2 u + \cos u \sin u$

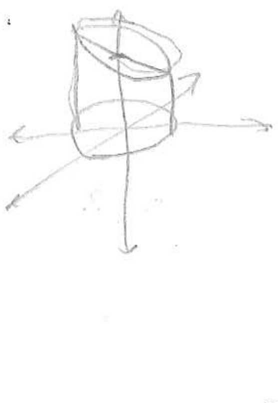
$\int_0^{2\pi} \cos^2 u$

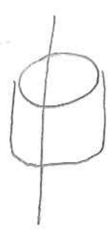
sin → 0 → 2π same

0

what does this mean? it means we take the surface and add all x to it?

The surface integral $\iint_S x \, dS$ is (circle one) $\boxed{\text{negative} \quad \text{zero} \quad \text{positive}}$.





← is this 0 or not?

Circle the correct answers.

4. (3 points) Let $\boldsymbol{F}$ be a vector field on an open connected domain $\mathcal{D}$ with continuous second order partial derivatives. Which of the following statements are always true, and which are only true if $\mathcal{D}$ is simply connected?

   (a) If $\mathrm{curl}(\boldsymbol{F}) = 0$ then $\boldsymbol{F}$ is conservative.

   Always true — Only true if $\mathcal{D}$ is simply connected

   (b) If $\boldsymbol{F}$ has a potential function then $\boldsymbol{F}$ is conservative.

   Always true — Only true if $\mathcal{D}$ is simply connected

   (c) If $\boldsymbol{F}$ is conservative then $\mathrm{curl}(\boldsymbol{F})$ is zero.

   Always true — Only true if $\mathcal{D}$ is simply connected

5. (5 points) Which of the following statements is true for all vector fields, and which is true only for conservative vector fields?

   (a) The line integral along a path from P to Q does not depend on which path is chosen.

   True for all vector fields — Only true for conservative vector fields

   (b) The line integral around a closed curve is zero.

   True for all vector fields — Only true for conservative vector fields

   (c) The line integral over an oriented curve C does not depend on how C is parametrized as long as each parametrization preserves the orientation of C.

   True for all vector fields — Only true for conservative vector fields

   (d) The (vector) line integral is equal to the (scalar) line integral of the tangential component along the curve.

   True for all vector fields — Only true for conservative vector fields

   (e) The line integral changes sign if the orientation of the curve is reversed.

   True for all vector fields — Only true for conservative vector fields

$\int \langle x,y\rangle \cdot r'(t) \overset{?}{=} \int f(r(t)) \|r(t)\|$

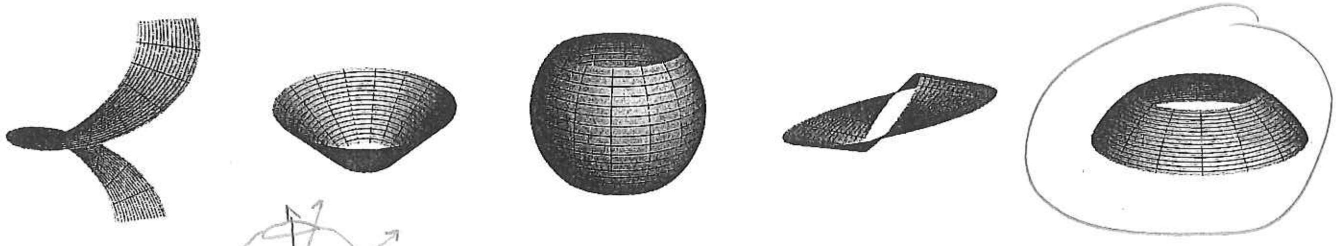
6. (12 points) Multiple choice. Circle the correct answer.

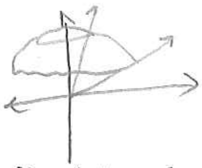
(a) Consider the vector field $\boldsymbol{F} = \langle xz, e^z - yz, \cos x\rangle$. Is there a function f such that $\boldsymbol{F} = \nabla f$?

Yes   No   We don't have enough information

$\frac{\partial F_1}{\partial y} = 0 \quad \frac{\partial F_2}{\partial x} = 0 \quad \frac{\partial F_3}{\partial x} = 0 \quad \frac{\partial F}{\partial z} = x$

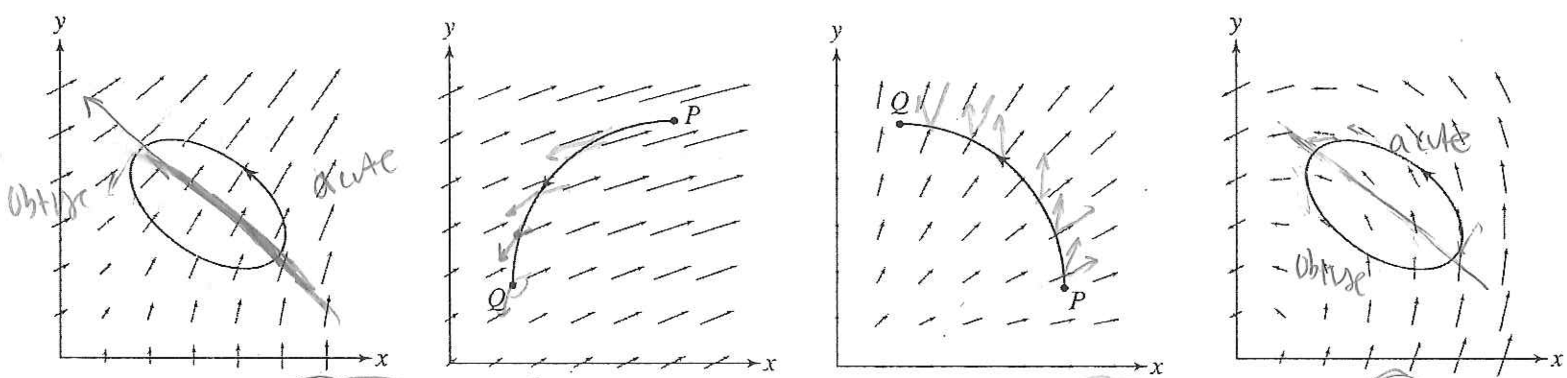
(b) $G(\theta, \phi) = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi\rangle$ with $0 \le \theta \le 2\pi$ and $\frac{\pi}{6} \le \phi \le \frac{\pi}{3}$ parametrizes which of the surfaces below? (circle one)





(c) Consider the line integrals $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for the vector fields $\boldsymbol{F}$ and paths $\boldsymbol{r}$ below.
Exactly **two** of the line integrals are zero, **one** is positive, and the remaining **one** is negative.
Circle "negative", "zero", or "positive" below each picture to indicate your answers.

Note: the closed curves are oriented counterclockwise and the others are oriented $P \to Q$.



negative zero positive   negative zero positive   negative zero positive   negative zero positive

(d) $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ is zero if (circle one)

$\boldsymbol{F}$ is tangent to S at every point $\boldsymbol{F}$ is perpendicular to S at every point