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Section	3	B
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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Spherical coordinates:

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

This derivative might be useful:

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

Page:	1	2	3	4	5	Total
Points:	10	10	10	8	12	50
Score:						

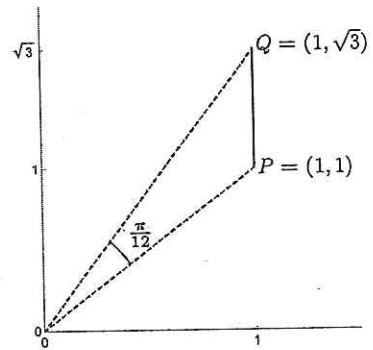
You may use this page for scratch work.

1. (10 points) Let F denote the vortex field $F = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Let C be the straight line segment from $P = (1, 1)$ to $Q = (1, \sqrt{3})$ (see the picture).

Show that $\int_C F \cdot dr = \frac{\pi}{12}$.

$$\begin{aligned} r(t) &= (1, t) \\ r'(t) &= (0, 1) \\ F(r(t)) &= \left\langle \frac{-t}{1+t^2}, \frac{1}{1+t^2} \right\rangle \\ F(r(t)) \cdot r'(t) &= \frac{1}{1+t^2} \\ \int_C F \cdot dr &= \int_1^{\sqrt{3}} \frac{1}{1+t^2} dt \\ &= \left[\tan^{-1} t \right]_1^{\sqrt{3}} \\ &= \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{\pi}{12} \quad // \text{ shown.} \end{aligned}$$



(b) Suppose that C is the ellipse parametrized by $r(t) = \langle 5 \cos(2t), 2 \sin(2t) \rangle$ for $0 \leq t \leq \pi$.

Compute $\int_C F \cdot dr$. Box your answer.

F is conservative when domain does not include $(0,0)$

~~$$\begin{aligned} r'(t) &= \langle -10 \sin 2t, 4 \cos 2t \rangle \\ F(r(t)) &= \left\langle \frac{-2 \sin 2t}{25 \cos^2 2t + 4 \sin^2 2t}, \frac{5 \cos 2t}{25 \cos^2 2t + 4 \sin^2 2t} \right\rangle \\ F \cdot r'(t) &= \frac{20 \sin^2 2t + 20 \cos^2 2t}{25 \cos^2 2t + 4 \sin^2 2t} \\ &= \frac{20}{25 \cos^2 2t + 4 \sin^2 2t} \end{aligned}$$~~

~~$$F = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$~~

$$\begin{aligned} \frac{F_1}{dy} &= \frac{-1}{(x^2+y^2)} + \frac{-y(2y)}{(x^2+y^2)^2} \\ &= \frac{-x^2+y^2}{(x^2+y^2)^2} \end{aligned}$$

~~$$\frac{F_2}{dx} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$~~

$$= \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{F_1}{dy} \quad // \text{ shown.}$$

~~$$\int \frac{-y}{x^2+y^2} dx = \tan^{-1} \frac{x}{y} + C$$

$$\int \frac{x}{x^2+y^2} dy = \tan^{-1} \frac{y}{x} + C$$

$$f = \tan^{-1} \frac{x}{y}$$~~

~~$$\begin{aligned} \int_C F \cdot dr &= f(r(\pi)) - f(r(0)) \\ &= \tan^{-1} 0 - \tan^{-1} 0 \\ &= \boxed{0} \end{aligned}$$~~

(c) Is the vortex field F conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.

Yes... F is conservative if and only if it fulfills the path $\frac{F_1}{dy} = \frac{F_2}{dx}$

No. The domain $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected, so F is only conservative when its domain is a simply connected domain, which it fulfills in the domain $\mathbb{R}^2 \setminus \{(0,0)\}$

2. (10 points) You do not need to simplify your answers.

(a) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the straight line from $P = (1, 2, 3)$ to $Q = (4, 5, 6)$ and

$$\mathbf{F} = \left\langle \frac{2xy}{x^2+z}, \ln(x^2+z), \frac{y}{x^2+z} \right\rangle.$$

Box your answer

$$(1+t)^2 + 3+t = 1+2t+t^2+3+t = t^2+3t+4$$

$$\frac{\partial F_2}{\partial z} = \frac{1}{x^2+z} = \frac{\partial F_3}{\partial y}$$

$$\frac{\partial F_3}{\partial x} = \frac{2y}{(x^2+z)^2} = \frac{\partial F_1}{\partial z}$$

$$\frac{\partial F_1}{\partial x} = \frac{2y}{x^2+z} = \frac{\partial F_2}{\partial y}$$

$$\mathbf{r}(t) = (1, 2, 3) + t(3, 3, 3)$$

$$\mathbf{r}(t) = (1+t, 2+t, 3+t)$$

$$\mathbf{r}'(t) = \langle 1, 1, 1 \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle \frac{2(1+t)(2+t)}{(1+t)^2+(3+t)}, \ln((1+t)^2+(3+t)), \frac{2+t}{(1+t)^2+(3+t)} \right\rangle$$

$\therefore \mathbf{F}$ is conservative on a simply connected domain.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \frac{2xy}{x^2+z} dx = y \ln x^2+z + C$$

$$\int \ln(x^2+z) dy = y \ln(x^2+z) + C$$

$$\int \frac{y}{x^2+z} dz = y \ln(x^2+z) + C$$

$C=0$

$$f = y \ln(x^2+z)$$

From P to Q , \mathbf{F} is continuous.

(b) Suppose that $\mathbf{F} = \langle y, z, 0 \rangle$ and that S is the surface parametrized by

$$\mathbf{G}(u, v) = (u^3 - v, u + v, v^2), \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 3.$$

$$= 5 \ln 22 - 2 \ln 4$$

with downward-pointing normal.

Fill in the limits and integrand of the integral below so that it equals $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$$\mathbf{T}_u = \langle 3u^2, 1, 0 \rangle$$

$$\mathbf{T}_v = \langle -1, 1, 2v \rangle$$

$$\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$$

$$= \langle 2v, -6u^2v, 3u^2+1 \rangle$$

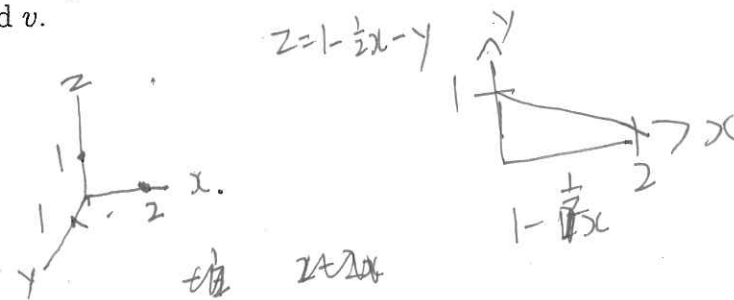
$$\mathbf{F}(\mathbf{G}(u, v)) = \langle u+v, v^2, 0 \rangle$$

$$\mathbf{F} \cdot \mathbf{N} = 2uv + 2v^2 - 6u^2v^3$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (2uv + 2v^2 - 6u^2v^3) du dv$$

You do not need to show work on this page.

3. (10 points) (a) Give a parametrization $G : D \rightarrow S$, where S is the triangle in \mathbb{R}^3 with vertices $(2, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ in the plane $\frac{1}{2}x + y + z = 1$. Be sure to explicitly specify the domain D and call your parameters u and v .

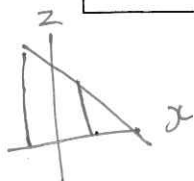


$$D = \left\{ 0 \leq u \leq 2, \quad 0 \leq v \leq 1 - \frac{1}{2}u \right\}$$

$$G(u, v) = \left\langle u, v, 1 - \frac{1}{2}u - v \right\rangle$$

- (b) Let S be the portion of the cylinder $x^2 + y^2 = 1$ between the xy -plane and the plane $x + z = 2$. We parametrize S by $G(u, v) = \langle \cos u, \sin u, v \rangle$, where (fill these in)

$$0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2 - \cos u$$



Handwritten notes: $z \geq 0$, $z \leq 2 - x$

The surface integral $\iint_S x \, dS$ is (circle one) negative zero positive

Handwritten integral: $\int_0^{2\pi} \int_0^{2-\cos u} \cos u \, dv \, du$
 $2\cos u - \cos^2 u$

Circle the correct answers.

4. (3 points) Let \mathbf{F} be a vector field on an open connected domain \mathcal{D} with continuous second order partial derivatives. Which of the following statements are always true, and which are only true if \mathcal{D} is simply connected?

(a) If $\text{curl}(\mathbf{F}) = 0$ then \mathbf{F} is conservative.

Always true

Only true if \mathcal{D} is simply connected

(b) If \mathbf{F} has a potential function then \mathbf{F} is conservative.

Always true

Only true if \mathcal{D} is simply connected

(c) If \mathbf{F} is conservative then $\text{curl}(\mathbf{F})$ is zero.

Always true

Only true if \mathcal{D} is simply connected

5. (5 points) Which of the following statements is true for all vector fields, and which is true only for conservative vector fields?

(a) The line integral along a path from P to Q does not depend on which path is chosen.

True for all vector fields

Only true for conservative vector fields

(b) The line integral around a closed curve is zero.

True for all vector fields

Only true for conservative vector fields

(c) The line integral over an oriented curve \mathcal{C} does not depend on how \mathcal{C} is parametrized as long as each parametrization preserves the orientation of \mathcal{C} .

True for all vector fields

Only true for conservative vector fields

(d) The (vector) line integral is equal to the (scalar) line integral of the tangential component along the curve.

True for all vector fields

Only true for conservative vector fields

(e) The line integral changes sign if the orientation of the curve is reversed.

True for all vector fields

Only true for conservative vector fields

$\frac{1}{2}xz$ $\frac{1}{2}x^2z + C$ $ye^z - \frac{1}{2}y^2z + C$
 $+ 2\cos x$

6. (12 points) Multiple choice. Circle the correct answer.

(a) Consider the vector field $F = \langle xz, e^z - yz, \cos x \rangle$. Is there a function f such that $F = \nabla f$?

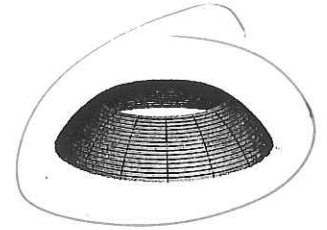
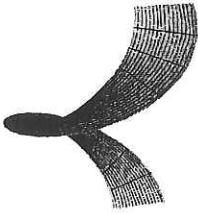
Yes

No

We don't have enough information



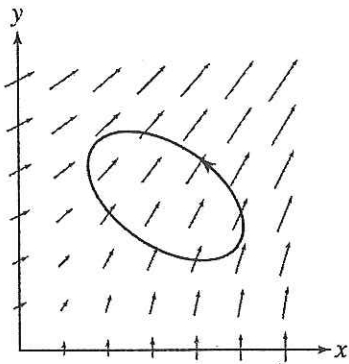
(b) $G(\theta, \phi) = \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle$ with $0 \leq \theta \leq 2\pi$ and $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$ parametrizes which of the surfaces below? (circle one)



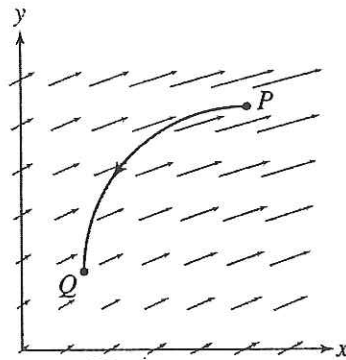
(c) Consider the line integrals $\int_c F \cdot dr$ for the vector fields F and paths r below.

Exactly two of the line integrals are zero, one is positive, and the remaining one is negative. Circle "negative", "zero", or "positive" below each picture to indicate your answers.

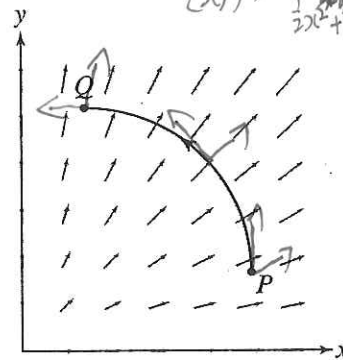
Note: the closed curves are oriented counterclockwise and the others are oriented $P \rightarrow Q$.



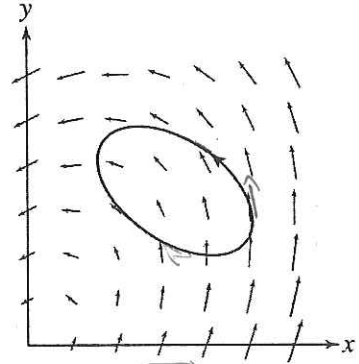
negative **zero** positive



negative zero positive



negative zero **positive**



negative **zero** positive

(d) $\iint_S F \cdot dS$ is zero if (circle one)

F is tangent to S at every point

F is perpendicular to S at every point

