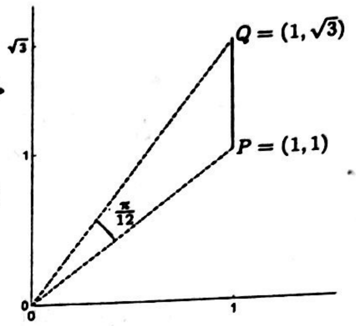
points) Let $\boldsymbol{F}$ denote the vortex field $\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

) Let C be the straight line segment from $P=(1,1)$ to $Q=(1,\sqrt{3})$ (see the picture).
Show that $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{\pi}{12}$.



The vortexfield is in a simply connected domain when $x \neq 0$, in this case, it sufeices.
In simply connected domain, F is conservative as it has the potential function $F = \nabla f = \theta$

$\therefore \int_C F \cdot dr = f(Q) - f(P) = \tan^{-1}(\sqrt{3}) - \tan^{-1}(1) = \frac{\pi}{3} - \frac{\pi}{4}$
$= \frac{\pi}{12}$

$r(t) = \langle 1, t \rangle$
$r'(t) = \langle 0, 1 \rangle$

$r'(t) \cdot F(r(t))$
$= \frac{1}{1+t^2} \cdot 1 = \frac{1}{1+t^2} = (1+t^2)^{-1}$

~~$\int_C F \cdot dr =$~~

) Suppose that C is the ellipse parametrized by $\boldsymbol{r}(t) = \langle 5\cos(2t), 2\sin(2t) \rangle$ for $0 \le t \le \pi$.
Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$. Box your answer. $\because x = 5\cos(2t)$ is not equal to zero ex

$r'(t) = \langle -10\sin 2t, 4\cos 2t \rangle$

$F = \left\langle \frac{-2\sin(2t)}{25\cos^2(2t) + 4\sin^2(2t)}, \frac{5\cos(2t)}{25\cos^2(2t) + 4\sin^2 2t} \right\rangle$

~~$\int_C F \cdot dr$~~ $= F$

$F \cdot r'(t) = \frac{20\sin^2(2t)}{4 + 21\cos^2(2t)} + \frac{20\cos^2 2t}{4 + 21\cos^2(2t)} = \frac{20}{4 + 21\cos^2 2t}$

$r(\pi) = \langle 5\cos 2\pi, 0 \rangle = \langle 5, 0 \rangle$
$r(0) = \langle 5, 0 \rangle$

$\int_C F \cdot dr = f(\pi) - f(0) = \boxed{\cancel{0}}$ as ~~it is conservative~~.

) Is the vortex field $\boldsymbol{F}$ conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.

Yes. Let $f = \theta = \tan^{-1}\frac{y}{x}$

$\nabla f = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = F$ for all points except for $x=0$. $(0,0)$

Therefore. F has a potential function in this domain.
So it is conservative.

+1

2. (10 points) You do not need to simplify your answers.

(a) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where C is the straight line from $P = (1,2,3)$ to $Q = (4,5,6)$ and

$$\boldsymbol{F} = \left\langle \frac{2xy}{x^2+z}, \ln(x^2+z), \frac{y}{x^2+z} \right\rangle.$$

$y(x^2+z)^{-1}$ $\frac{}{x^2+z} \cdot 2x$

Box your answer

~~$\frac{\partial F_1}{\partial y} = \frac{2x}{x^2+z}$ $\frac{\partial F_2}{\partial x} = 2x \cdot (-1)(x^2+z)^{-2}$~~

$\because \frac{\partial F_1}{\partial z} = \frac{\partial\, 2xy(x^2+z)^{-1}}{\partial z} = -2xy(x^2+z)^{-2}$

5

$f = \int F_1 dx + \psi(y,z) = \int F_2 dy + \Phi(x,z) = \int F_3 dz + g(x,y)$

$\because \int F_2 dy = \ln(x^2+z)\, y + c = A$

$\because \frac{\partial A}{\partial x} = \frac{2xy}{x^2+z} = F_1$, $\frac{\partial A}{\partial z} = \frac{y}{x^2+z} = F_3$, $\frac{\partial A}{\partial y} = \ln(x^2+z) = F_2$

$\therefore f = \ln(x^2+z)\,y$

$\therefore \int_C F \cdot dr = f(Q) - f(P)$

$= \ln(16+6) \cdot 5 - \ln(1+3) \cdot 2$

$= \boxed{5\ln 22 - 2\ln 4}$

(b) Suppose that $\boldsymbol{F} = \langle y, z, 0 \rangle$ and that S is the surface parametrized by

$$G(u,v) = (u^3 - v, u+v, v^2), \quad 0 \le u \le 2, \quad 0 \le v \le 3.$$

with downward-pointing normal.

simplify

Fill in the limits and integrand of the integral below so that it equals $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

$T_u = (3u^2, 1, 0)$

$T_v = \langle -1, 1, 2v \rangle$

$N = T_u \times T_v = \begin{vmatrix} i & j & k \\ 3u^2 & 1 & 0 \\ -1 & 1 & 2v \end{vmatrix} = \langle 2v, -(6u^2v), 3u^2+1 \rangle$

$F(G(u,v)) = \langle u+v, v^2, 0 \rangle$

$F \cdot N = 2uv + 2v^2, -6u^2v^3 = 2uv + 2v^2 - 6u^2v^3$

4

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_0^3 \int_0^2 2uv + 2v^2 - 6u^2v^3 \quad du\, dv$$

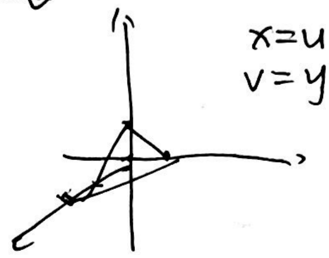
You do not need to show work on this page.

1. (10 points) (a) Give a parametrization $G : D \to S$, where S is the triangle in $\mathbf{R}^3$ with vertices $(2,0,0)$, $(0,1,0)$, $(0,0,1)$ in the plane $\frac{1}{2}x + y + z = 1$. Be sure to explicitly specify the domain D and call your parameters u and v.

~~$\frac{1}{2}x = 1 - y - z$~~

$x = 2 - 2y - 2z$

$z = 1 - \frac{1}{2}x - y$



$x = u$

$v = y$

3

$D = \{ \; 0 \leq u \leq 2 \;,\; 0 \leq v \leq \cancel{1} \; \}$

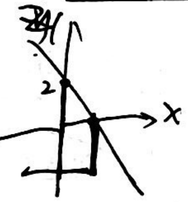
$G(u,v) = \langle \; u \;,\; v \;,\; 1 - \frac{1}{2}u - v \; \rangle$

(b) Let S be the portion of the cylinder $x^2 + y^2 = 1$ between the xy-plane and the plane $x + z = 2$. We parametrize S by $G(u,v) = \langle \cos u, \sin u, v \rangle$, where (fill these in)

$z = 2 - x$

$z = 2 - \cos u$

$0 \leq u \leq 2\pi \;,\; 0 \leq v \leq 2 - \cos u$

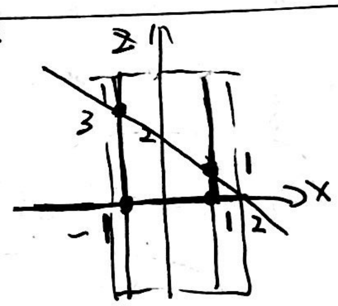


$x+z$

$z = 2 - x$ $\qquad$ $x + z = 2$

$\cos u + v = 2$

~~$\cos u + \sin u = 0$~~



3

$\cos u < \cos u$ $\qquad$ $z \geq 2 - x$

The surface integral $\iint_S x \, dS$ is (circle one) negative zero positive

$\frac{1+\sin 2\theta}{2}$

~~$T_v = T_u = 1$~~

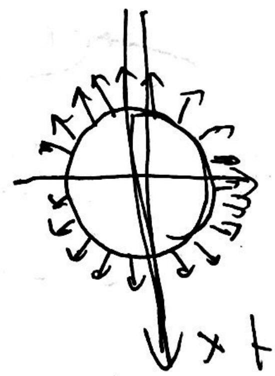
$T_u = \langle -\sin u, \cos u, 0 \rangle$

$T_v = \langle 0, 0, 1 \rangle$

$\begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$\langle \cos u, \sin u, 0 \rangle$ $\quad$ $\sim \cos u$

$\int \cos^2 u + \cancel{\cos u} \sin u.$



$x+$

Circle the correct answers.

4. (3 points) Let $\boldsymbol{F}$ be a vector field on an open connected domain $\mathcal{D}$ with continuous second order partial derivatives. Which of the following statements are always true, and which are only true if $\mathcal{D}$ is simply connected?

(a) If $\mathrm{curl}(\boldsymbol{F}) = 0$ then $\boldsymbol{F}$ is conservative.

Always true　　　Only true if $\mathcal{D}$ is simply connected

(b) If $\boldsymbol{F}$ has a potential function then $\boldsymbol{F}$ is conservative.

Always true　　　Only true if $\mathcal{D}$ is simply connected

(c) If $\boldsymbol{F}$ is conservative then $\mathrm{curl}(\boldsymbol{F})$ is zero.

Always true　　　Only true if $\mathcal{D}$ is simply connected

5. (5 points) Which of the following statements is true for all vector fields, and which is true only for conservative vector fields?

(a) The line integral along a path from P to Q does not depend on which path is chosen.

True for all vector fields　　　Only true for conservative vector fields

(b) The line integral around a closed curve is zero.

True for all vector fields　　　Only true for conservative vector fields

(c) The line integral over an oriented curve $\mathcal{C}$ does not depend on how $\mathcal{C}$ is parametrized as long as each parametrization preserves the orientation of $\mathcal{C}$.

True for all vector fields　　　Only true for conservative vector fields

(d) The (vector) line integral is equal to the (scalar) line integral of the tangential component along the curve.

True for all vector fields　　　Only true for conservative vector fields

(e) The line integral changes sign if the orientation of the curve is reversed.

True for all vector fields　　　Only true for conservative vector fields

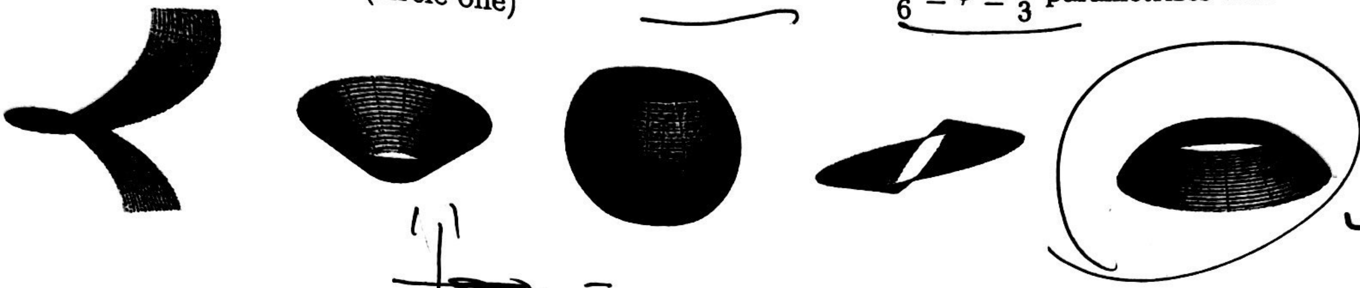
(12 points) Multiple choice. Circle the correct answer.

(a) Consider the vector field $\boldsymbol{F} = \langle xz, e^z - yz, \cos x\rangle$. Is there a function f such that $\boldsymbol{F} = \nabla f$?

Yes   No   We don't have enough information

$\frac{F_1}{\partial z} = x$ $\frac{F_3}{\partial x} = -\sin x$

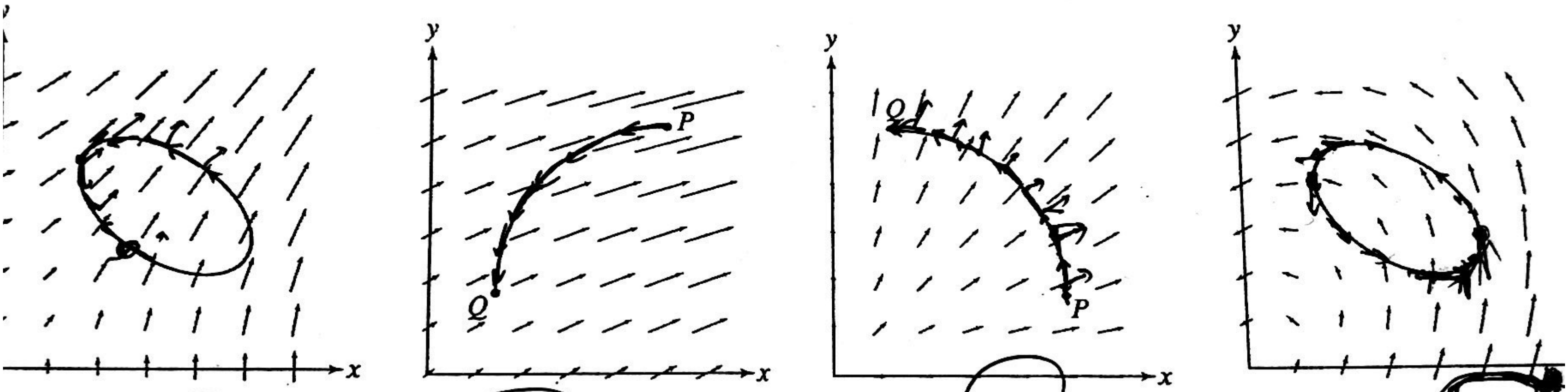
(b) $G(\theta, \phi) = \langle \cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$ with $0 \le \theta \le 2\pi$ and $\frac{\pi}{6} \le \phi \le \frac{\pi}{3}$ parametrizes which of the surfaces below? (circle one)



π

(c) Consider the line integrals $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for the vector fields $\boldsymbol{F}$ and paths $\boldsymbol{r}$ below.
Exactly **two** of the line integrals are zero, **one** is positive, and the remaining **one** is negative.
Circle "negative", "zero", or "positive" below each picture to indicate your answers.

Note: the closed curves are oriented counterclockwise and the others are oriented $P \to Q$.



negative zero positive   negative zero positive   negative zero positive   negative zero positive

$F \cdot \|N\| = 0$ if $N \perp F$ $F \perp N, N \perp S$
$F \parallel S$

(d) $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ is zero if (circle one)

$\boldsymbol{F}$ is tangent to S at every point $\boldsymbol{F}$ is perpendicular to S at every point