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Section	3	D

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- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
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Spherical coordinates:

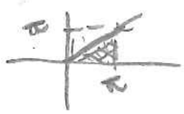
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta$$

Page:	1	2	3	4	5	6	7	8	Total
Points:	12	12	12	12	12	15	15	10	100
Score:									

1. (4 points) Evaluate the integral $\int_0^\pi \int_y^\pi \frac{\sin x}{x}\,dx\,dy$ by changing the order of integration.

$0 \le y \le \pi \quad y \le x \le \pi$



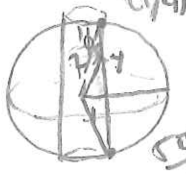
$$\int_0^\pi \int_y^\pi \frac{\sin x}{x}\,dx\,dy = \int_0^\pi \int_0^x \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^\pi \frac{\sin x}{x} y \Big]_0^x dx = \int_0^\pi \sin x\,dx = -\cos x \Big]_0^\pi = \boxed{2}$$

2. (8 points) Using spherical coordinates, set up *but do not evaluate* a triple integral that computes the volume of a sphere of radius 2 from which a central cylinder of radius 1 has been removed.

$(1,y) \to (1,\sqrt{3})$

$y = \sqrt{3}$



$\frac{5\pi}{6}$

$$\phi = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\ell \le \rho \le 2 \qquad \sin\phi = \frac{1}{\ell} \to \ell = \frac{1}{\sin\phi} = \csc\phi$$

$$\boxed{\int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{\csc\phi}^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$

3. (12 points) Let $\boldsymbol{F}$ denote the vortex field $\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Suppose that C_R is the circle of radius R centered at $(0,0)$ oriented counterclockwise. **By parametrizing** C_R, compute $\oint_{C_R} \boldsymbol{F} \cdot d\boldsymbol{r}$. Box your answer

Note: you may not use the fundamental theorem of line integrals or anything about the winding number in this problem. Also, the answer is not zero.

$G(t) = (R\cos t, R\sin t) \qquad G'(t) = (-R\sin t, R\cos t)$

$\oint_{C_R} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle \cdot \langle -R\sin t, R\cos t \rangle \, dt = \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt =$

$\int_0^{2\pi} 1 \, dt = \boxed{2\pi}$

(b) Compute $\text{curl}_z(\boldsymbol{F})$. Show your work. Box your answer

$\text{curl}_z(F) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) =$

$(x^2+y^2)^{-1} \cdot 1 + x \cdot -1(x^2+y^2)^{-2} \cdot 2x - \left((x^2+y^2)^{-1} \cdot -1 + (-y) \cdot -1(x^2+y^2)^{-2} \cdot 2y\right) =$

$(x^2+y^2)^{-1} - 2x^2(x^2+y^2)^{-2} + (x^2+y^2)^{-1} - 2y^2(x^2+y^2)^{-2} = 2(x^2+y^2)^{-1} - (2x^2+2y^2)(x^2+y^2)^{-2} =$

$\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2} = \frac{2}{x^2+y^2} - \frac{2}{x^2+y^2} = \boxed{0}$

(c) Fill in the blanks:

(i) If $\boldsymbol{F} = \nabla f$ on a domain D then $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} =$ ___0___ for every closed curve C in D.

(ii) If $\text{curl}_z(\boldsymbol{F}) = 0$ on a ___simply connected___ domain D then $\boldsymbol{F}$ is conservative.

(d) Is the vortex field $\boldsymbol{F}$ conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.

No. Loop integrals in the domain of F that go around the origin don't equal 0.

4. (12 points) Let $\boldsymbol{F} = \langle 2x, 0, -2z \rangle$.

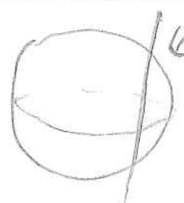
(a) Verify that $\boldsymbol{A} = \langle yz, -xz, yx \rangle$ is a vector potential for $\boldsymbol{F}$.

$F = \text{curl}(A)$?

$$\langle 2x, 0, -2z \rangle = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & -xz & yx \end{bmatrix} = \langle x+x, -(y-y), -z-z \rangle$$

$$= \langle 2x, 0, -2z \rangle \checkmark$$

(b) Let S be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \leq 3$, oriented with outward-pointing normal vector. Find the flux of $\boldsymbol{F}$ through S. *Hint:* use the result of part (a). Box your answer



$\leftarrow x=3 \to 3^2 + y^2 + z^2 = 13$

$y^2 + z^2 = 4 \to$

$r(t) = \langle 0, 2\cos t, 2\sin t \rangle$

(opposite direction)

$r'(t) = \langle 0, -2\sin t, 2\cos t \rangle$

$$\iint_S \text{curl}(\vec{A}) \cdot d\vec{S} = \oint_{\partial S} \vec{A} \cdot d\vec{r} = -\int_0^{2\pi} \langle 4\cos t \sin t, 0, 0 \rangle \cdot \langle 0, -2\sin t, 2\cos t \rangle \, dt =$$

$$-\int_0^{2\pi} 0 \, dt = \boxed{0}$$

$A = \nabla f$ where $f = x^2 - z^2$

lol I'm dumb A is conservative, $\oint A \cdot d\vec{r} = 0 \checkmark$

(c) Let S' be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \geq 3$, oriented with outward-pointing normal vector. Find the flux of $\boldsymbol{F}$ through S'. Box your answer
You don't need to show your work for this part of the problem.

$\boxed{\text{Flux of } F = 0}$

5. (12 points) Given that C is a simple closed curve in the plane $x+y+z=1$ (oriented counterclockwise when viewed from above) that encloses a surface area of 5, compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for $\boldsymbol{F} = \langle 3z, 2x, 4y \rangle$.

Box your answer *Hint:* it may be helpful to remember that $\iint_S \boldsymbol{G} \cdot d\boldsymbol{S} = \iint_S (\boldsymbol{G} \cdot \boldsymbol{n}) dS$.

$N = \langle 1, 1, 1 \rangle \quad n = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$$\text{Curl}(\vec{F}) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z & 2x & 4y \end{bmatrix} = \langle 4-0, -(0-3), 2-0 \rangle$$

$$= \langle 4, 3, 2 \rangle$$

$C = \partial S$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_S \langle 4, 3, 2 \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle \, dS =$$

$$\iint_S \frac{4}{\sqrt{3}} + \frac{3}{\sqrt{3}} + \frac{2}{\sqrt{3}} \, dS = \iint_S \frac{9}{\sqrt{3}} \, dS = 3\sqrt{3} \iint_S 1 \, dS = 3\sqrt{3}(5) = \boxed{15\sqrt{3}}$$

6. (12 points) Let $\boldsymbol{F} = \langle z^2 x, \frac{1}{3}y^3 + \sin^2 z, x^2 z + y^2\rangle$.

(a) Let $\mathcal{D}$ be the unit disk $x^2+y^2 \le 1$ in the xy-plane, oriented downward. Compute $\iint_{\mathcal{D}} \boldsymbol{F}\cdot d\boldsymbol{S}$.

It may be helpful to know that $\int_0^{2\pi} \sin^2\theta \, d\theta = \pi$. $\boxed{\text{Box your answer}}$
Hint: if $\mathcal{D}$ is parametrized via $G(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ then $\boldsymbol{N} = \pm\langle 0,0,r\rangle$.

$$\iint_D \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \langle 0, \tfrac{1}{3}(r\sin\theta)^3, (r\sin\theta)^2\rangle \cdot -\langle 0,0,r\rangle \, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^1 -r^3\sin^2\theta \, dr\, d\theta = \int_0^{2\pi} -\tfrac{1}{4}\sin^2\theta \, d\theta = -\tfrac{1}{4}\int_0^{2\pi}\sin^2\theta\, d\theta =$$

$$-\tfrac{1}{4}(\pi) = \boxed{\frac{-\pi}{4}}$$

(b) Let $\mathcal{S}$ be the top half of the sphere $x^2+y^2+z^2 = 1$, oriented upward. Compute $\iint_{\mathcal{S}} \boldsymbol{F}\cdot d\boldsymbol{S}$.

$\boxed{\text{Box your answer}}$ *Hint:* you should use your answer to part (a). If you cannot do part (a), let A denote the value of the integral in part (a) and give your answer in terms of A.

$$\iint_S F\cdot dS + \iint_D F\cdot dS = \iiint_W \operatorname{div}(\vec{F})\, dV = \iiint_W z^2+y^2+x^2 \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta =$$

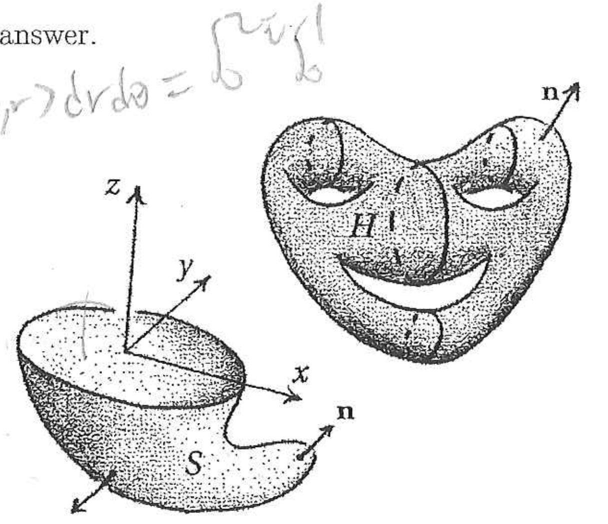
$$2\pi \int_0^{\pi/2} \left.\tfrac{1}{5}\rho^5 \sin\phi\right]_0^1 d\phi = 2\pi\int_0^{\pi/2} \tfrac{1}{5}\sin\phi\, d\phi = \tfrac{2\pi}{5}\left[-\cos\phi\right]_0^{\pi/2} = \tfrac{2\pi}{5}$$

$$\iint_S F\cdot dS = \tfrac{2\pi}{5} - \iint_D F\cdot dS = \tfrac{2\pi}{5} - \left(-\tfrac{\pi}{4}\right) = \boxed{\frac{2\pi}{5} + \frac{\pi}{4}}$$

7. (15 points) Multiple choice. Circle the correct answer.

$\int_0^{2\pi}\int_0^1 \langle x,y,z\rangle\cdot\langle 0,0,r\rangle\,dr\,d\theta = \int_0^{2\pi}\int_0^1$

Let S and H be the surfaces shown to the right. The boundary of S is the unit circle in the xy-plane, while H has no boundary. Let $\boldsymbol{G} = \langle x, y, z\rangle$.



(a) The flux $\iint_H \boldsymbol{G}\cdot d\boldsymbol{S}$ is

negative zero positive

(b) The flux $\iint_S \boldsymbol{G}\cdot d\boldsymbol{S}$ is

negative zero positive

Hint for (b): use the divergence theorem.

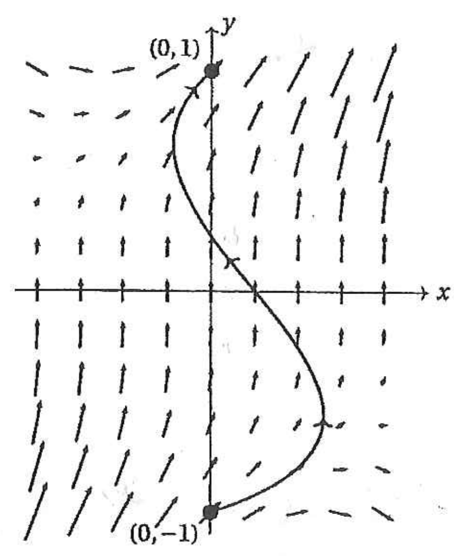
(c) The flux $\iint_S \operatorname{curl}\boldsymbol{G}\cdot d\boldsymbol{S}$ is

negative zero positive

(d) A vector field is shown to the right. For scale, $\boldsymbol{F}(0,0) = \langle 0, 0.1\rangle$.

Given that <u>$\boldsymbol{F}$ is conservative</u>, estimate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$, where C is the curve shown from $(0,-1)$ to $(0,1)$.

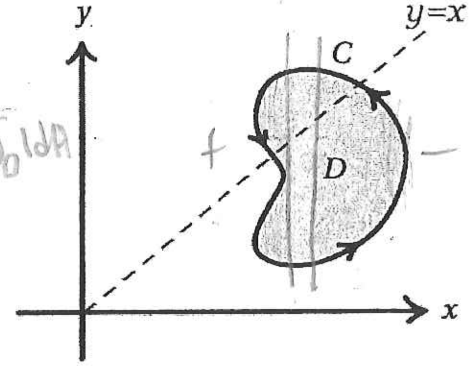
-0.5 -0.2 0 0.2 0.5



(e) Which of the following statements makes sense and is true for any vector field $\boldsymbol{F}$ in $\mathbb{R}^3$ whose components have continuous second-order partial derivatives?

$\nabla(\operatorname{curl}\boldsymbol{F}) = 0$ $\operatorname{div}(\operatorname{curl}\boldsymbol{F}) = 0$ $\operatorname{div}(\nabla\boldsymbol{F}) = 0$ $\operatorname{curl}(\operatorname{curl}\boldsymbol{F}) = 0$

8. (15 points) Multiple choice. Circle the correct answer.
Consider the region D in the plane bounded by the curve C as shown to the right. For parts (a)–(c), circle the best answer.



(a) For $\boldsymbol{F}(x, y) = \langle x+1, y^2 \rangle$, the integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is

negative   zero   positive

(b) The integral $\int_C (-y\,dx + 2dy)$ is $= \iint_D \mathrm{curl}_2(\vec{F})\cdot dA = \iint_D 1\,dA$

negative   zero   positive

(c) The integral $\iint_D (y - x)\, dA$ is

negative   zero   positive

Hint for (c): look at the location of D in the plane.

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the plane sending the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ to the triangle with vertices $(0,0)$, $(1,2)$, $(-1,3)$, respectively. Find the Jacobian of T.

$A = \frac{1}{2}$

1   2   3   4   5

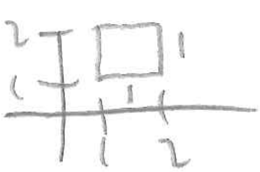
$A = 2\cdot 3 - \frac{1}{2}(1)(3) - \frac{1}{2}(1)(2) - \frac{1}{2}(1)(2) = 6 - \frac{3}{2} - 1 - 1 = \frac{5}{2}$

(e) Let $\mathcal{R} = [1,2] \times [1,2]$ and let $\mathcal{D} = G(\mathcal{R})$, where G is the map $G(u, v) = (u^2/v, v^2/u)$. Compute the area of $\mathcal{D}$.

1   2   3   4   5

$\mathrm{Jac}(G) = \det \begin{bmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v^2}{u^2} & \frac{2v}{u} \end{bmatrix} = \left(\frac{2u}{v}\right)\left(\frac{2v}{u}\right) - \left(\frac{-v^2}{u^2}\right)\left(\frac{-u^2}{v^2}\right) =$

$4 - 1 = 3$



9. (10 points) Fill in the blanks in the big theorems of vector calculus.

The fundamental theorem of line integrals. If C is an oriented curve from P to Q in $\mathcal{D}$ then

$$\int_C \nabla f \cdot dr = f(Q) - f(P).$$

Green's theorem. Let $\mathcal{D}$ be a domain whose boundary ∂D is a simple closed curve, oriented $\boxed{\text{such that if you are a normal vector walking along the boundary your}}$. Then
left hand points toward the domain

$$\iint_{\mathcal{D}} \boxed{\mathrm{curl}_z(\vec{F})}\, dA = \oint_{\boxed{\partial D}} \boldsymbol{F} \cdot dr.$$

or

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Stokes' theorem. Let S be a "sufficiently nice" surface, and let $\boldsymbol{F}$ be a vector field whose components have continuous partial derivatives on an open region containing S. Then

$$\iint_S \mathrm{curl}(\boldsymbol{F}) \cdot dS = \boxed{\oint_{\partial S} \vec{F} \cdot d\vec{r}}.$$

The integral on the right-hand side is defined relative to the boundary orientation of ∂S.

The divergence theorem. Let S be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Assume that S is piecewise smooth and is oriented by normal vectors pointing $\boxed{\text{outwards}}$. Let $\boldsymbol{F}$ be a vector field whose domain contains $\mathcal{W}$. Then

$$\boxed{\iiint_{\mathcal{W}} \mathrm{div}(\vec{F})\, dV} = \iint_{\partial \mathcal{W}} \boldsymbol{F} \cdot dS.$$

You may use this page for scratch work.