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Section	3	C

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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Spherical coordinates:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta$$

Page:	1	2	3	4	5	6	7	8	Total
Points:	12	12	12	12	12	15	15	10	100
Score:									

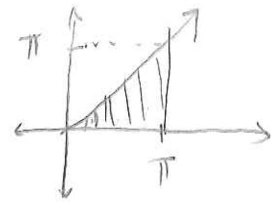
1. (4 points) Evaluate the integral $\int_0^{\pi} \int_y^{\pi} \frac{\sin x}{x}\, dx\, dy$ by changing the order of integration.

$\int_0^{\pi}$

$y \le x \le \pi$

$0 \le y \le \pi$

lower bound



$$\int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\, dy\, dx$$

$0 \le y \le x$

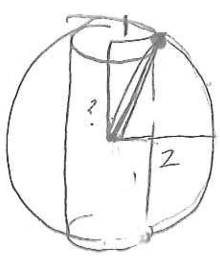
$0 \le x \le \pi$

$$\int_0^{\pi} \frac{\sin x}{\cancel{x}} \cancel{(x)}\, dx$$

$-\cos(\pi) + \cos(0)$

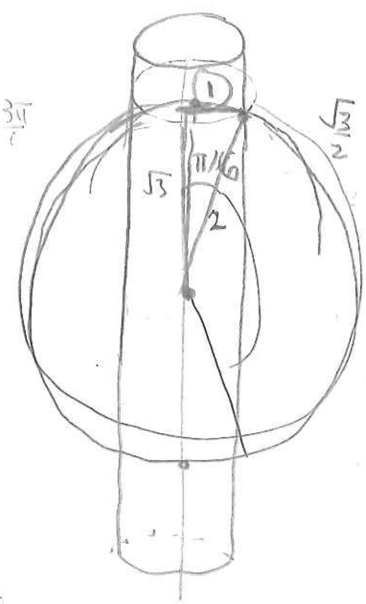
$-(-1) + 1 = \boxed{2}$

2. (8 points) Using spherical coordinates, set up *but do not evaluate* a triple integral that computes the volume of a sphere of radius 2 from which a central cylinder of radius 1 has been removed.



$1 \le \rho \le 2$

$\frac{\pi}{6} + \frac{2\pi}{6} + \frac{3\pi}{6}$



$\sqrt{3}$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2}$$

$x^2 + y^2 = 1$

$\frac{\pi}{6} \le \phi \le \frac{5\pi}{6}$

$0 \le \theta \le 2\pi$

$1 \le \rho \le 2$

$$\boxed{\int_{\pi/6}^{5\pi/6}\int_0^{2\pi}\int_1^{2} 1\, dV}$$

(in the order $d\rho\, d\theta\, d\phi$)

so dV can be replaced w/

$(\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi)$

3. (12 points) Let $\boldsymbol{F}$ denote the vortex field $\boldsymbol{F} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$.

(a) Suppose that C_R is the circle of radius R centered at $(0,0)$ oriented counterclockwise. **By parametrizing** C_R, compute $\displaystyle\oint_{C_R} \boldsymbol{F} \cdot d\boldsymbol{r}$. Box your answer

Note: you may not use the fundamental theorem of line integrals or anything about the winding number in this problem. Also, the answer is not zero.

$r(r,\theta) = (R\cos\theta, R\sin\theta)$ $\quad dr = \langle -R\sin\theta, R\cos\theta \rangle$

$F = \left\langle -\dfrac{R\sin\theta}{R^2}, \dfrac{R\cos\theta}{R^2} \right\rangle$ $\quad F \cdot dr = \sin^2\theta + \cos^2\theta$

$\displaystyle\int_0^{2\pi} 1 \, d\theta = \boxed{2\pi}$

$F = \left\langle -\dfrac{\sin\theta}{R}, \dfrac{\cos\theta}{R} \right\rangle$

(b) Compute $\text{curl}_z(\boldsymbol{F})$. Show your work. Box your answer

Work on scratch paper

$\text{curl}_z(F)$ $\quad \partial x \; \partial y \; \partial z$ $\quad \partial F_1 \; \partial F_2$

$\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$ $\quad \boxed{0}$

$\dfrac{\partial}{\partial y} \dfrac{x}{x^2+y^2} \partial x =$

$\dfrac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$

(c) Fill in the blanks:

(i) If $\boldsymbol{F} = \nabla f$ on a domain D then $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} =$ ___0___ for every closed curve C in D.

(ii) If $\text{curl}_z(\boldsymbol{F}) = 0$ on a ___simply connected___ domain D then $\boldsymbol{F}$ is conservative.

(d) Is the vortex field $\boldsymbol{F}$ conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.

~~No, because the surfaces can include (0,0) and even tho the domain of the vortex field does not~~ (pt a).

No, the domain has a hole at (0,0) making it not a simply connected domain, and surfaces can include that point at (0,0), as in pt a, and $\oint F \cdot dr \neq 0$ anymore.

$\partial x \ \partial y \ \partial z$ ✓

4. (12 points) Let $\boldsymbol{F} = \langle 2x, 0, -2z \rangle$.

(a) Verify that $\boldsymbol{A} = \langle yz, -xz, yx \rangle$ is a vector potential for $\boldsymbol{F}$.

curl(A) = F if A is a vector potential

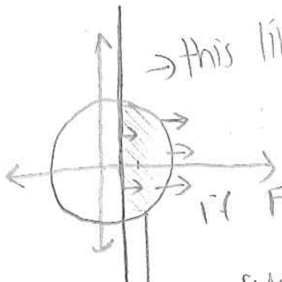
$$\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ yz & -xz & yx \end{vmatrix} \quad (x+x), -(y-y), -z-z)$$

$$\langle -2x, 0, -2z \rangle \checkmark$$

(b) Let $\mathcal{S}$ be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \leq 3$, oriented with outward-pointing normal vector. Find the flux of $\boldsymbol{F}$ through $\mathcal{S}$. *Hint:* use the result of part (a). Box your answer

$r^2 = 13$

$r = \sqrt{13}$ ...



→ this line...

$\oint_b F \cdot dr$

if $F = \cancel{\#}$ curl (A)

surface independent

just need boundary ~~to be the same~~

chose plane enclosed by $z=0$, $x^2+y^2+z^2=13$, $x \leq 3$

$\iint F \cdot ds = \oint A \cdot dr$

~~closed curve~~

circle of $x^2+y^2=13$

and $x \leq 3$

$(0,0)$ $y\cancel{x}$

$\boxed{0}$

but also: $F = \langle 2x, 0, -2z \rangle$

$f = x^2 - z^2$

$\nabla f = F$,

so closed curves $= 0$.

$r = (r\cos\theta, r\sin\theta, 0)$

$r'(t)$

$A \cdot dr$ $\quad a = (0, 0, yx) \cdot (-r\sin\theta, r\cos\theta, 0)$

$= (0, 0)$

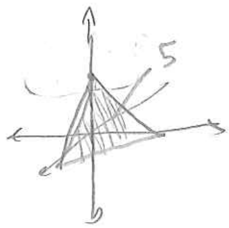
(c) Let $\mathcal{S}'$ be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \geq 3$, oriented with outward-pointing normal vector. Find the flux of $\boldsymbol{F}$ through $\mathcal{S}'$. Box your answer
You don't need to show your work for this part of the problem.

$\boxed{0}$ also a closed boundary

5. (12 points) Given that C is a simple closed curve in the plane $x+y+z=1$ (oriented counterclockwise when viewed from above) that encloses a surface area of 5, compute $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ for $\boldsymbol{F}=\langle 3z, 2x, 4y\rangle$.

Box your answer *Hint:* it may be helpful to remember that $\iint_S \boldsymbol{G}\cdot d\boldsymbol{S} = \iint_S (\boldsymbol{G}\cdot\boldsymbol{n})dS$.

$\int_C F\cdot dr \quad F=\langle 3z, 2x, 4y\rangle$



$\int_C F\cdot dr = \iint_S \text{curl} F \, dA$

$$\begin{matrix} i & j & k \\ \partial x & \partial y & \partial z \\ 3z & 2x & 4y \end{matrix}$$

$\langle 4, 3, 2\rangle$

$\iint_S \langle 4,3,2\rangle\cdot\langle 1,1,1\rangle$

$=\iint_S 4+3+2\, dS$

$= 9\underbrace{\iint_S 1\, dS}_{\text{area of surface}}$

$= \boxed{45}$

parameterized:

$x=x \quad G(x,y)=(x, y, 1-x-y)$

$y=y$

$z=1-x-y$

$T_x=\langle 1, 0, -1\rangle$

$T_y=\langle 0, 1, -1\rangle$

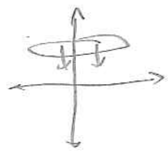
$N=\langle (-1)(-1), -(-1), 1\rangle = \langle 1, 1, 1\rangle$

$z^2 + y^2 +$

6. (12 points) Let $\boldsymbol{F} = \langle z^2 x, \frac{1}{3}y^3 + \sin^2 z, x^2 z + y^2 \rangle$.

(a) Let $\mathcal{D}$ be the unit disk $x^2 + y^2 \le 1$ in the xy-plane, oriented downward. Compute $\iint_{\mathcal{D}} \boldsymbol{F} \cdot d\boldsymbol{S}$.

It may be helpful to know that $\int_0^{2\pi} \sin^2\theta \, d\theta = \pi$. Box your answer
Hint: if $\mathcal{D}$ is parametrized via $G(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ then $\boldsymbol{N} = \pm\langle 0,0,r\rangle$.



$G(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$

normal vector includes jacobian

$N = \langle 0, 0, -r\rangle$ $\quad z = 0$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$x = r\cos\theta$

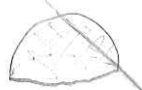
$y = r\sin\theta$

$F \cdot N = \langle 0, \text{doesn't matter}, r^2\sin^2\theta\rangle \cdot \langle 0, 0, -r\rangle$

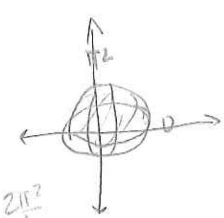
$-\pi \int_0^1 r^3 \, dr = -\pi\left[\frac{r^4}{4}\right]_0^1 \leftarrow \int_0^1\int_0^{2\pi} -r^3\sin^2\theta \, d\theta \, dr = \boxed{-\frac{\pi}{4}}$

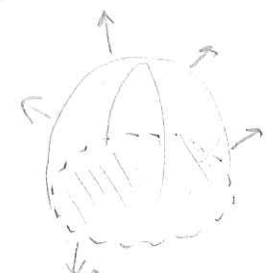
(b) Let $\mathcal{S}$ be the top half of the sphere $x^2 + y^2 + z^2 = 1$, oriented upward. Compute $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$.

Box your answer *Hint:* you should use your answer to part (a). If you cannot do part (a), let A denote the value of the integral in part (a) and give your answer in terms of A.



~~surface independent?~~





$\iiint_W \operatorname{div} F \cdot dV = \iint_{\partial W} F \cdot dS$

$\frac{4\pi^2}{3}$ $\quad \frac{2\pi^2}{5}$

$\iint_{\partial W} F \cdot ds = 0$

closed surface

$\langle z^2, y^2, x^2\rangle$

Work on on scratch paper

$z^2 + y^2 + x^2$

$\iint_D F \cdot dS + \iint_S F \cdot ds = 0$

ans =

$r\sin\phi\,(r^2 + \cos\phi^2)$

$-\frac{\pi}{4} + \frac{\pi}{4} = 0 \quad \iint_S F \cdot ds = \frac{\pi}{4}$

$2\pi \; \frac{?}{2}\left[r^3\right]_0$

$\boxed{\frac{2\pi}{5} + \frac{\pi}{4}}$

$\partial x \ \partial y \ \partial z$

$x^2 \ y^2 \ z^2$

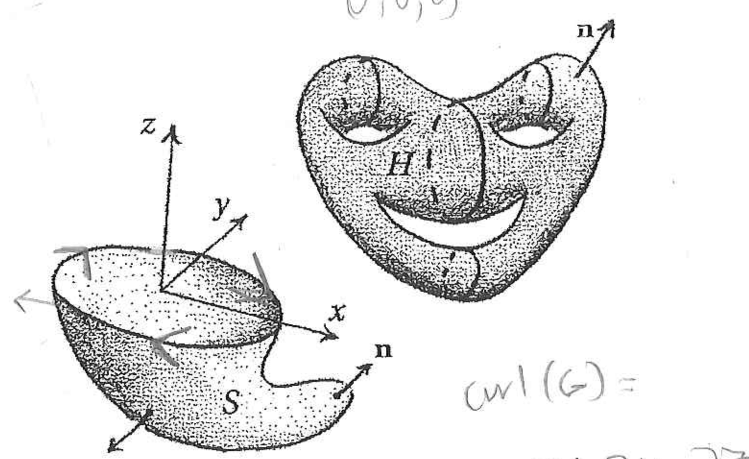
$(0,0,0)$

7. (15 points) Multiple choice. Circle the correct answer.

$G(x,y,z)$

can area be neg?

Let S and H be the surfaces shown to the right. The boundary of S is the unit circle in the xy-plane, while H has no boundary. Let $\boldsymbol{G} = \langle x, y, z \rangle$.



$\operatorname{div} G(x,y,z) = 3$

$\operatorname{curl}(G) =$

$\partial x \ \partial y \ \partial z$

$x \ y \ z$

$(0,0,0)$

(a) The flux $\iint_H \boldsymbol{G} \cdot d\boldsymbol{S}$ is

negative zero positive

(b) The flux $\iint_S \boldsymbol{G} \cdot d\boldsymbol{S}$ is

negative zero positive

Hint for (b): use the divergence theorem.

$\frac{\partial}{\partial x}\left(\frac{F_2}{\partial z} - \frac{F_3}{\partial y}\right)$

(c) The flux $\iint_S \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S}$ is

negative zero positive

(x^2, y^2, z^2)

$\oint_{\partial S} F \cdot dr$

$(0,1)$

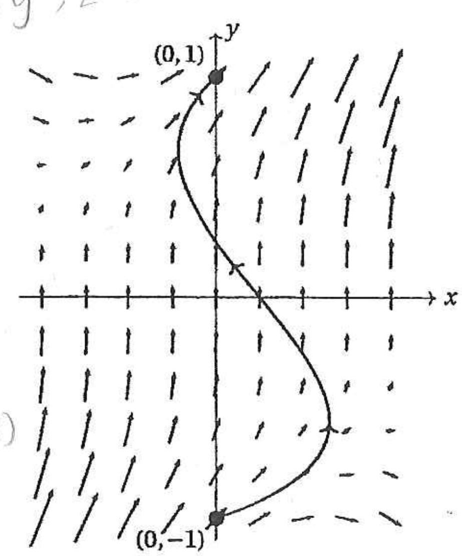
(d) A vector field is shown to the right. For scale, $\boldsymbol{F}(0,0) = \langle 0, 0.1 \rangle$.

$r(y) = (0, y)$

$\int_{-1}^{1} y \, dx$

Given that <u>$\boldsymbol{F}$ is conservative</u>, estimate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, $\int_{-1}^{1} 0.1 \, dy$

where C is the curve shown from $(0,-1)$ to $(0,1)$.



-0.5 $\quad -0.2$ $\quad 0$ $\quad 0.2$ $\quad 0.5$

$(0.1)(1) - (-1)(0.1)$

x does not change

∂x has no effect

when is curl(F) 0

(e) Which of the following statements makes sense and is true for any vector field $\boldsymbol{F}$ in $\mathbb{R}^3$ whose components have continuous second-order partial derivatives?

$\nabla(\operatorname{curl} \boldsymbol{F}) = 0$ $\quad \operatorname{div}(\operatorname{curl} \boldsymbol{F}) = 0$ $\quad \cancel{\operatorname{div}(\nabla \boldsymbol{F}) = 0}$ $\quad \cancel{\operatorname{curl}(\operatorname{curl} \boldsymbol{F}) = 0}$

$\frac{\partial}{\partial x}$

$(\nabla) \to \operatorname{curl} F \to$ div

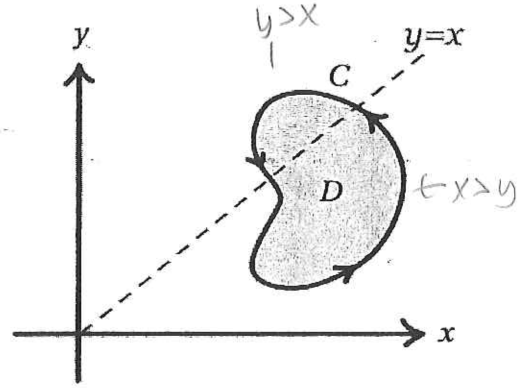
div = $\nabla \cdot \nabla \cdot f$ take partial twice

not always 0

$\frac{x^2}{2} + x + \frac{y^3}{3}$

8. (15 points) Multiple choice. Circle the correct answer.

Consider the region D in the plane bounded by the curve C as shown to the right. For parts (a)–(c), circle the best answer.



(a) For $\boldsymbol{F}(x, y) = \langle x + 1, y^2 \rangle$, the integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is

negative zero positive

$(-y, 2)$

(b) The integral $\int_C (-y dx + 2dy)$ is

$+)$

negative zero positive

(c) The integral $\iint_D (y - x)\, dA$ is

negative zero positive

Hint for (c): look at the location of D in the plane.

$u, v = 0, 0$ A B C D

$(1, 0)$ $G(u,v) = \langle u - v,\ 2u + 3v \rangle$

$$\begin{array}{c|cc} & u & v \\ \hline \nabla x & 1 & -1 \\ \nabla y & 2 & 3 \end{array}$$

$3 - (-2)$

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the plane sending the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ to the triangle with vertices $(0, 0)$, $(1, 2)$, $(-1, 3)$, respectively. Find the Jacobian of T.

$\sqrt{9}$ $\sqrt{1^2}$ 1 2 3 4 5

$(-1,3)$ $\sqrt{3}$ AD − CB $\begin{vmatrix} A & B \\ C & D \end{vmatrix}$ $\langle B, D \rangle$ $\langle A, C \rangle$ $\langle -1, 3 \rangle$ $\langle 1, 2 \rangle$

$\sqrt{5} \cdot \sqrt{5} \cdot \sqrt{2} = \frac{5\sqrt{2}}{2}$

$\frac{1}{2} \cdot 5$ (e) Let $\mathcal{R} = [1, 2] \times [1, 2]$ and let $\mathcal{D} = G(\mathcal{R})$, where G is the map $G(u, v) = (u^2/v, v^2/u)$. Compute the area of $\mathcal{D}$.

$3^2 + 1^2 = \sqrt{10} \cdot \sqrt{5}$ $\frac{5\sqrt{2}}{2}$

1 2 3 4 5

4 $\frac{2u}{v}\left(\frac{2v}{u}\right) -$

Jacobian:

$$\begin{array}{c|ccc} & u & v & \\ \hline x & \frac{2u}{v} & u^2 \cdot \frac{-1}{v^2} & \frac{u^2}{v^2} \\ y & v^2 \cdot \frac{1}{u^2} & \frac{2v}{u} & \end{array}$$

-1

$(1,1) \to (1, 1)$ $(1,1)$

$(1,2) \to (\frac{1}{2}, 4)$

$(2,1)$ $(4, 1/2)$

$(2,2)$ $(2, 2)$

5

4

$4 - (+1) = 3$

$\frac{1}{u}$ u^{-1} $\to (-)u^{-2}$

9. (10 points) Fill in the blanks in the big theorems of vector calculus. ✓

The fundamental theorem of line integrals. If C is an oriented curve from P to Q in $\mathcal{D}$ then

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(Q) - f(P).$$

Green's theorem. Let $\mathcal{D}$ be a domain whose boundary ∂D is a simple closed curve, oriented $\boxed{\text{counterclockwise}}$. Then

$$\iint_{\mathcal{D}} \boxed{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}\, dA = \oint_{\boxed{\partial D}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

$\partial x \quad \partial y \quad \partial z$

$F \quad F_2$

$\uparrow$

(curl_z)

Stokes' theorem. Let $\mathcal{S}$ be a "sufficiently nice" surface, and let $\boldsymbol{F}$ be a vector field whose components have continuous partial derivatives on an open region containing $\mathcal{S}$. Then

$$\iint_{\mathcal{S}} \operatorname{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} = \boxed{\oint_{\partial S} F \cdot dr}.$$

The integral on the right-hand side is defined relative to the boundary orientation of $\partial\mathcal{S}$.

The divergence theorem. Let $\mathcal{S}$ be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing $\boxed{\text{otward}}$. Let $\boldsymbol{F}$ be a vector field whose domain contains $\mathcal{W}$. Then

$$\boxed{\iiint_W \operatorname{div} F \cdot dV} = \iint_{\partial\mathcal{W}} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

You may use this page for scratch work.

$$\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

$$\left\langle \frac{-y}{R}, \frac{x}{R} \right\rangle$$

$\cancel{\frac{\partial R}{\partial x} \frac{x}{R}}$

$$\frac{\partial r}{\partial x} \frac{x}{(x^2+y^2)} = \frac{x}{r}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$\frac{(x^2+y^2)\cdot x(2x)}{(x^2+y^2)^2} - \frac{-(x^2+y^2)\cdot 2y^2}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(2x^2+2y^2)}{x^2+y^2} = 2$$

$1 - 1 = 0$

let $x^2 + y^2 = R$

3b) $\text{curl}_z(F)$ [redacted]

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$\frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} - \left(\frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2} \right)$$

$$\frac{(x^2+y^2) - 2x^2 + x^2+y^2 - 2y^2}{(x^2+y^2)^2}$$

$$\frac{2x^2+2y^2-2x^2-2y^2}{(x^2+y^2)^2} = \boxed{0}$$

~~∇ curl F~~

$\partial x \quad \partial y \quad \partial z$

$5xy^2,\ 3yz^2,\ 2z^2x$

~~$\langle x^3, y^3, z^3 \rangle$~~

~~$\partial x \quad \partial y \quad \partial z$~~

~~$x^3y^3z^3,\ x^3y^3z^3,\ x^3y^3z^3$~~

$\partial x \quad \partial y \quad \partial z$

$\langle -3y^2,\ 2z^2,\ -5x^2 \rangle$

~~$\partial x \quad \partial y \quad \partial z$~~

~~$x^3y^3z^3,\ x^3y^3z^3,\ x^3y^3z^2$~~

$0 \quad 0 \quad 0$

$x^3z^3y^2 - x^3y^3z^2,$

$5xy,\ 3yz,\ 2zx$

$5y,$