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Section	3	B

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- Fill out your name, section letter, and UID above.
- Do not open this exam packet until you are told that you may begin.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
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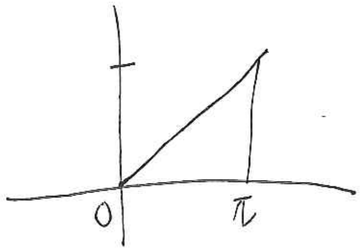
Spherical coordinates:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta$$

Page:	1	2	3	4	5	6	7	8	Total
Points:	12	12	12	12	12	15	15	10	100
Score:									

1. (4 points) Evaluate the integral $\displaystyle\int_0^{\pi}\int_y^{\pi}\frac{\sin x}{x}\,dx\,dy$ by changing the order of integration.



$$\cancel{\int_0^{\pi}\int_{x}^{\pi}\frac{\sin x}{x}\,dy\,dx}$$

$$\cancel{= \int_0^{\pi}\left[\frac{\sin x}{x}\,y\right]_x^{\pi}}$$

$$\cancel{= \int_0^{\pi}\frac{\sin x}{x}\pi - \sin x\,dx}$$

$$\int_0^{\pi}\int_0^{x}\frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^{\pi}\left[\frac{\sin x}{x}y\right]_0^{x}\,dx$$

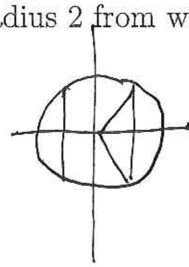
$$= \int_0^{\pi}\sin x\,dx$$

$$= \left[-\cos x\right]_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$= 2$$

2. (8 points) Using spherical coordinates, set up *but do not evaluate* a triple integral that computes the volume of a sphere of radius 2 from which a central cylinder of radius 1 has been removed.



when $x=1$, $y=0$,

$x^2+y^2+z^2=4$

$1+ \; z^2=3$

$z=\sqrt{3}$

angle from x axis $= \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$

$r = \rho\sin\phi$

$\rho\sin\phi \geq 1$

$\rho \geq \operatorname{cosec}\phi$

$$\int_0^{2\pi}\int_{\frac{5\pi}{6}}^{\frac{\pi}{6}}\int_{\operatorname{cosec}\phi}^{2}\rho^2\sin\phi\,dr\,d\phi\,d\theta$$

3. (12 points) Let $\boldsymbol{F}$ denote the vortex field $\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Suppose that C_R is the circle of radius R centered at $(0,0)$ oriented counterclockwise. **By parametrizing** C_R, compute $\oint_{C_R} \boldsymbol{F} \cdot d\boldsymbol{r}$. Box your answer

Note: you may not use the fundamental theorem of line integrals or anything about the winding number in this problem. Also, the answer is not zero.

$r(t) = \langle R\cos t, R\sin t \rangle$

$r'(t) = \langle -R\sin t, R\cos t \rangle$

$F(r(t)) = \left\langle -\frac{R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle$

$F(r(t)) \cdot r'(t) = \sin^2 t + \cos^2 t$

$= 1$

$\int_0^{2\pi} 1 \, dt = \boxed{2\pi}$

(b) Compute $\text{curl}_z(\boldsymbol{F})$. Show your work. Box your answer

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-1}{(x^2+y^2)} + \frac{y(2y)}{((x^2+y^2))^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}F_1 - \frac{\partial}{\partial x}F_2 = \frac{-2(x^2+y^2)+2y^2+2x^2}{(x^2+y^2)^2}$$

$$= \boxed{0}$$

(c) Fill in the blanks:

(i) If $\boldsymbol{F} = \nabla f$ on a domain D then $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} =$ $\underline{\ 0\ }$ for every closed curve C in D.

(ii) If $\text{curl}_z(\boldsymbol{F}) = 0$ on a $\underline{\text{simply connected}}$ domain D then $\boldsymbol{F}$ is conservative.

(d) Is the vortex field $\boldsymbol{F}$ conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.

No. The domain $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected

$T_\theta = \langle \sqrt{13}\sin\phi\sin\theta, \sqrt{13}\sin\phi\cos\theta, 0\rangle$

$T_\phi = \langle \sqrt{13}\cos\phi\cos\theta, \sqrt{13}\cos\phi\sin\theta, -\sqrt{13}\sin\phi\rangle$

$N = \langle 13\sin^2\phi\cos\theta, 13\sin^2\phi\sin\theta, -13\sin\phi\cos\phi\rangle$

$2(13)^{1.5}\sin^3\phi\cos\theta \quad 2(13)^{1.5}\sin\phi\cos^2\phi$

4. (12 points) Let $\boldsymbol{F} = \langle 2x, 0, -2z\rangle$.

(a) Verify that $\boldsymbol{A} = \langle yz, -xz, yx\rangle$ is a vector potential for $\boldsymbol{F}$.

$$\operatorname{curl}(A) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & yx \end{vmatrix}$$

$$= \langle x+x, -y+y, -z-z\rangle$$

$$= \langle 2x, 0, -2z\rangle = F \quad // \text{ shown}$$

(b) Let S be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \leq 3$, oriented with outward-pointing normal vector. Find the flux of $\boldsymbol{F}$ through S. *Hint:* use the result of part (a). Box your answer

Boundary of S is the circle $y^2+z^2=4$ when $x=0$

$$\oint_{\partial S} A\,dr = \iint_S F\,ds$$

when $x=3$, $y^2+z^2=4$

$$r(t) = \langle 3, 2\cos t, 2\sin t\rangle$$

$$r'(t) = \langle 0, -2\sin t, 2\cos t\rangle$$

$$A(r(t)) = \langle 4\sin t\cos t, -6\sin t, 6\cos t\rangle$$

$$A(r(t))\cdot r'(t) = 12\sin^2 t + 12\cos^2 t = 12$$

$$\cancel{\int_{2\pi}^{0} 12\sin^2 t + 12\cos^2 t\,dt = \left[-12\cos t + 12\sin t\right]_{2\pi}^{0} = -12+12 = 0}$$

$$\int_{2\pi}^{0} 12\,dt$$

$$= \boxed{-24\pi}$$

(c) Let S' be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \geq 3$, oriented with outward-pointing normal vector. Find the flux of $\boldsymbol{F}$ through S'. Box your answer
You don't need to show your work for this part of the problem.

$\boxed{-24\pi}$

By stoke's theorem, since S' and S share the same boundary and F has a vector potential, the flux through S and S' are the same.

5. (12 points) Given that C is a simple closed curve in the plane $x+y+z=1$ (oriented counterclockwise when viewed from above) that encloses a surface area of 5, compute $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ for $\boldsymbol{F} = \langle 3z, 2x, 4y\rangle$.

$\boxed{\text{Box your answer}}$ *Hint:* it may be helpful to remember that $\iint_S \boldsymbol{G}\cdot d\boldsymbol{S} = \iint_S (\boldsymbol{G}\cdot\boldsymbol{n})dS$.

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 2x & 4y \end{vmatrix}$$

$$= \langle 4, 3, 2\rangle$$

$$\int_C F\cdot dr = \iint_S \text{curl}(F)\, d S$$

$$= \iint_S \langle 4,3,2\rangle \cdot n\, dS$$

$$N = \langle 1,1,1\rangle$$

$$n = \left\langle \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right\rangle$$

$$= \iint \frac{4}{\sqrt3} + \frac{3}{\sqrt3} + \frac{2}{\sqrt3}\, dA$$

$$= 3\sqrt3\,(5)$$

$$\boxed{= 15\sqrt3}$$

6. (12 points) Let $\boldsymbol{F} = \langle z^2x, \frac{1}{3}y^3 + \sin^2 z, x^2z + y^2\rangle$.

(a) Let $\mathcal{D}$ be the unit disk $x^2 + y^2 \leq 1$ in the xy-plane, oriented downward. Compute $\iint_{\mathcal{D}} \boldsymbol{F} \cdot d\boldsymbol{S}$.

It may be helpful to know that $\int_0^{2\pi} \sin^2\theta \, d\theta = \pi$. Box your answer

Hint: if $\mathcal{D}$ is parametrized via $G(r, \theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ then $\boldsymbol{N} = \pm\langle 0, 0, r\rangle$.

z component of $F(G) = r^2\sin^2\theta$

$F(G)\cdot N = -r^3\sin^2\theta$

$\int_0^{2\pi}\int_0^1 -r^3\sin^2\theta \, dr\, d\theta$

$= \int_0^{2\pi} \left[-\frac{1}{4}r^4\sin^2\theta\right]_0^1 d\theta$

$= \int_0^{2\pi} -\frac{1}{4}\sin^2\theta \, d\theta$

$= \boxed{-\frac{1}{4}\pi}$

~~$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2x & \frac{1}{3}y^3+\sin^2 z & x^2z+y^2 \end{vmatrix}$~~

~~$= \langle 2y - 2\sin z\cos z, -2zx + 2zx, \ldots\rangle$~~

~~$= \langle 2y - 2\sin z\cos z, 0, 0\rangle$~~

~~$\operatorname{curl}(F)(G) = \langle 2r\sin\theta, 0, 0\rangle$~~

(b) Let S be the top half of the sphere $x^2 + y^2 + z^2 = 1$, oriented upward. Compute $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

Box your answer *Hint:* you should use your answer to part (a). If you cannot do part (a), let A denote the value of the integral in part (a) and give your answer in terms of A.

$\operatorname{div}(F) = z^2 + y^2 + x^2$

$\iint_S F\cdot dS = \iiint_W \operatorname{div}(F)\, dA - \iint_D F\cdot dS$ found above.

$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta + \frac{1}{4}\pi$

$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{5}\sin\phi \, d\phi\, d\theta + \frac{1}{4}\pi$

$= \frac{1}{5}\int_0^{2\pi} [\cos\phi]_0^{\frac{\pi}{2}} d\theta + \frac{1}{4}\pi$

$= \frac{1}{5}\int_0^{2\pi} +1 \, d\theta + \frac{1}{4}\pi$

$= \frac{2}{5}\pi + \frac{1}{4}\pi$

$= \boxed{\frac{13}{20}\pi}$

$G(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ $\quad \langle 0,0,r\rangle$ $\quad r(t) = (\cos t, \sin t, 0)$

$T_r = \langle \cos\theta, \sin\theta, 0\rangle$ $\quad r'(t) = (-\sin t, \cos t, 0)$

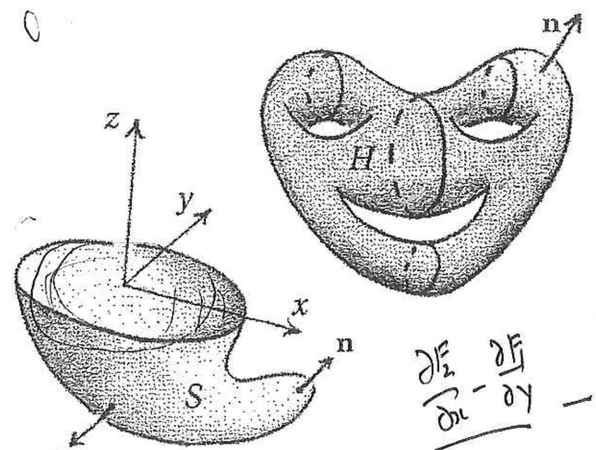
$T_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$ $\quad \langle \cos t, \sin t\rangle \cdot \langle -\sin t, \cos t, 0\rangle$

0

7. (15 points) Multiple choice. Circle the correct answer.

0

Let S and H be the surfaces shown to the right. The boundary of S is the unit circle in the xy-plane, while H has no boundary. Let $\boldsymbol{G} = \langle x, y, z\rangle$. $\quad$ div $F = 3$



$\dfrac{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}{\partial y} - \dfrac{\frac{\partial F_1}{\partial z} + \frac{\partial F_3}{\partial x}}{\partial z}$

(a) The flux $\displaystyle\iint_H \boldsymbol{G} \cdot d\boldsymbol{S}$ is

negative $\qquad$ zero $\qquad$ positive

(b) The flux $\displaystyle\iint_S \boldsymbol{G} \cdot d\boldsymbol{S}$ is

negative $\qquad$ zero $\qquad$ positive

Hint for (b): use the divergence theorem.

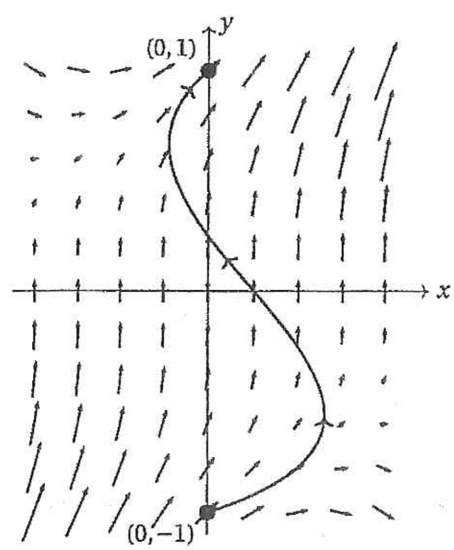
(c) The flux $\displaystyle\iint_S \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S}$ is

negative $\qquad$ zero $\qquad$ positive

(d) A vector field is shown to the right. For scale, $\boldsymbol{F}(0,0) = \langle 0, 0.1\rangle$.

Given that <u>$\boldsymbol{F}$ is conservative</u>, estimate $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is the curve shown from $(0,-1)$ to $(0,1)$.

$-0.5 \qquad -0.2 \qquad 0 \qquad 0.2 \qquad 0.5$



(e) Which of the following statements makes sense and is true for any vector field $\boldsymbol{F}$ in $\mathbb{R}^3$ whose components have continuous second-order partial derivatives?

$\operatorname{curl}(F) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right.$

$\nabla(\operatorname{curl} \boldsymbol{F}) = 0 \qquad \operatorname{div}(\operatorname{curl} \boldsymbol{F}) = 0 \qquad \operatorname{div}(\nabla \boldsymbol{F}) = 0 \qquad \operatorname{curl}(\operatorname{curl} \boldsymbol{F}) = 0$

$\operatorname{curl}(F) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$

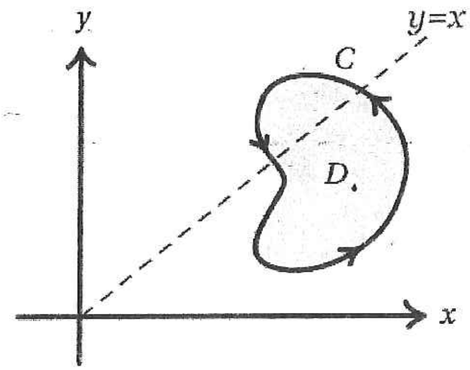
$\frac{u^3}{u} - v^3$

$\frac{1}{2}x^2 + x + \frac{1}{3}y^3$

$\langle -y, 2\rangle = 0 + 1$

8. (15 points) Multiple choice. Circle the correct answer.

Consider the region D in the plane bounded by the curve C as shown to the right. For parts (a)–(c), circle the best answer.



(a) For $F(x, y) = \langle x + 1, y^2 \rangle$, the integral $\int_C F \cdot dr$ is

negative   zero   positive

(b) The integral $\int_C (-y\,dx + 2dy)$ is $\langle -y, 2 \rangle$

negative   zero   positive

(c) The integral $\iint_D (y - x)\,dA$ is

negative   zero   positive

Hint for (c): look at the location of D in the plane.

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the plane sending the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ to the triangle with vertices $(0, 0)$, $(1, 2)$, $(-1, 3)$, respectively. Find the Jacobian of T.

1   2   3   4   5

$T(u,v) = (u - v, 2u + 3v)$

$Jac = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2$

(e) Let $\mathcal{R} = [1, 2] \times [1, 2]$ and let $\mathcal{D} = G(\mathcal{R})$, where G is the map $G(u, v) = (u^2/v, v^2/u)$. Compute the area of $\mathcal{D}$.

1   2   3   4   5

$x^2 y = \frac{u^4}{u} = u^3$

$y^2 x = \frac{v^4}{v} = v^3$

$Jac(G) = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix} = 4 - 1 = 3$

$\int_1^2 \int_1^2 3\,du\,dv$

9. (10 points) Fill in the blanks in the big theorems of vector calculus.

The fundamental theorem of line integrals. If $\mathcal{C}$ is an oriented curve from P to Q in $\mathcal{D}$ then

$$\int_{\mathcal{C}} \nabla f \cdot d\boldsymbol{r} = f(Q) - f(P).$$

Green's theorem. Let $\mathcal{D}$ be a domain whose boundary ∂D is a simple closed curve, oriented $\boxed{\text{with outward pointing Normal}}$. Then

$$\iint_{\mathcal{D}} \boxed{\cancel{F}\ \mathrm{curl}(F)}\, dA = \oint_{\boxed{\partial D}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Stokes' theorem. Let $\mathcal{S}$ be a "sufficiently nice" surface, and let $\boldsymbol{F}$ be a vector field whose components have continuous partial derivatives on an open region containing $\mathcal{S}$. Then

$$\iint_{\mathcal{S}} \mathrm{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} = \boxed{\oint_{\partial S} F \cdot dr}.$$

The integral on the right-hand side is defined relative to the boundary orientation of $\partial\mathcal{S}$.

The divergence theorem. Let $\mathcal{S}$ be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing $\boxed{\text{outwards}}$. Let $\boldsymbol{F}$ be a vector field whose domain contains $\mathcal{W}$. Then

$$\boxed{\iiint_W \mathrm{div}(F)\, dV} = \iint_{\partial \mathcal{W}} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

You may use this page for scratch work.