Math 32B Exam 2

Tian Yu Liu

TOTAL POINTS

49 / 50

QUESTION 1

Vortex Field 10 pts

1.1 Angle **4 / 4**

✓ + **2 pts** **Correct setup (parametrization and integral)**

✓ + **2 pts** **Correct evaluation of integral**

+ **4 pts** Using potential function arctan(y/x)

- **1 pts** Minor error

+ **0 pts** Incorrect

+ **1 pts** Attempting to parametrize using sines and cosines

1.2 Ellipse **3 / 3**

✓ - **0 pts** **Reasonable attempt**

- **3 pts** No reasonable attempt

1.3 Conservative **2 / 3**

✓ + **2 pts** **No**

+ **2 pts** Reasoning

+ **1 pts** Incorrect

+ **1 pts** Partially correct reasoning

QUESTION 2

Integrals 10 pts

2.1 Line integral **5 / 5**

✓ + **5 pts** **Correct**

+ **0 pts** Click here to replace this description.

+ **2 pts** Set up an integral with a correct parameterization

+ **2 pts** applied fundamental theorem

+ **2 pts** found a potential function

+ **1 pts** Correct answer

2.2 Surface Integral **4 / 5**

+ **5 pts** Correct

✓ + **2 pts** **Correct | normal vector |**

+ **1 pts** Correct orientation on normal

✓ + **2 pts** **Correct bounds**

+ **0 pts** Click here to replace this description.

QUESTION 3

Parametrizations 10 pts

3.1 Triangle **5 / 5**

✓ + **1 pts** **Correct inequality in D independent of the other variable.**

✓ + **2 pts** **Correct inequality in D dependent on the other variable.**

✓ + **2 pts** **Correct parameterisation of G.**

+ **0 pts** Incorrect

3.2 Cylinder **5 / 5**

✓ + **1 pts** **Correct u bounds.**

✓ + **2 pts** **Correct v bounds.**

✓ + **2 pts** **Correct choice (negative).**

+ **0 pts** Incorrect.

QUESTION 4

Simply connected 3 pts

4.1 a **1 / 1**

- **1 pts** Always True

✓ - **0 pts** **Only if D is simply connected**

- **1 pts** No answer

4.2 b **1 / 1**

✓ - **0 pts** **Always True**

- **1 pts** Only if D is SC

- **0 pts** No answer

4.3 c **1 / 1**

✓ **- 0 pts** **Always True**

- 1 pts Only true if D is SC

- 1 pts No answer

QUESTION 5

Conservative Vector Fields 5 pts

5.1 a **1 / 1**

- 1 pts All

✓ **- 0 pts** **Conservative**

5.2 b **1 / 1**

- 1 pts All

✓ **- 0 pts** **Conservative**

5.3 c **1 / 1**

✓ **- 0 pts** **All**

- 1 pts Conservative

5.4 d **1 / 1**

✓ **- 0 pts** **All**

- 1 pts Conservative

- 1 pts No Answer

5.5 e **1 / 1**

✓ **- 0 pts** **All**

- 1 pts Conservative

- 1 pts No Answer

QUESTION 6

MC 12 pts

6.1 a **3 / 3**

- 3 pts Yes

✓ **- 0 pts** **No**

- 3 pts Not enough info

- 3 pts No answer

6.2 b **3 / 3**

✓ **- 0 pts** **fifth choice**

- 3 pts any other choice

6.3 c **3 / 3**

✓ **+ 1 pts** **first is zero (max of 3 points for this problem)**

✓ **+ 1 pts** **second is negative**

✓ **+ 1 pts** **third is zero**

✓ **+ 1 pts** **fourth is positive**

- 1 pts too many zeros, negatives, or positives

+ 0 pts no answer

6.4 d **3 / 3**

✓ **- 0 pts** **F is tangent to S**

- 3 pts F is perpendicular to S

- 3 pts No answer

QUESTION 7

7 Bonus **1 / 0**

✓ **+ 1 pts** **Bonus point**

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3A	Ben Szczesny	T	GEOLOGY 4645
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3E	Ryan Wallace	T	BUNCHE 3156
3F		R	HAINES A25

Section	3	D

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- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time.
- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
- Quit working and close this packet when you are told to stop.

Spherical coordinates:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

This derivative might be useful:

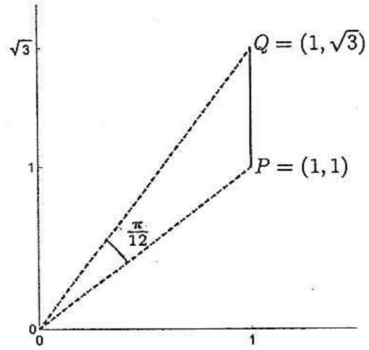
$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

Page:	1	2	3	4	5	Total
Points:	10	10	10	8	12	50
Score:						

You may use this page for scratch work.

1. (10 points) Let $\boldsymbol{F}$ denote the vortex field $\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Let C be the straight line segment from $P = (1,1)$ to $Q = (1, \sqrt{3})$ (see the picture). Show that $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{\pi}{12}$.



~~$r(t) = $ gradient~~

$r(t) = \langle 1, t \rangle, \ 1 \le t \le \sqrt{3}$

$r'(t) = \langle 0, 1 \rangle$

$\int_C F \cdot dr = \int_1^{\sqrt{3}} F(r(t)) \cdot r'(t)\, dt$

$= \int_1^{\sqrt{3}} \left\langle \frac{-t}{1+t^2}, \frac{1}{1+t^2} \right\rangle \cdot \langle 0, 1 \rangle \, dt$

$= \int_1^{\sqrt{3}} \frac{1}{1+t^2}\, dt = \left[\tan^{-1}(t)\right]_1^{\sqrt{3}} = \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{12}$

(b) Suppose that C is the ellipse parametrized by $\boldsymbol{r}(t) = \langle 5\cos(2t), 2\sin(2t) \rangle$ for $0 \le t \le \pi$. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$. Box your answer.

$r'(t) = \langle -10\sin(2t), 4\cos(2t) \rangle$

$\int_C F \cdot dr = \int_0^{\pi} F(r(t)) \cdot r'(t)\, dt = \int_0^{\pi} \left\langle -\frac{2\sin(2t)}{25\cos^2(2t) + 4\sin^2(2t)}, \frac{5\cos(2t)}{25\cos^2(2t)+4\sin^2(2t)} \right\rangle$

$\cdot \langle -10\sin(2t), 4\cos(2t) \rangle \, dt$

$= \int_0^{\pi} \frac{20\sin^2(2t) + 20\cos^2(2t)}{25\cos^2(2t) + 4\sin^2(2t)}\, dt$

$= \frac{20}{\pi}\int_0^{\pi} \frac{1}{25\cos^2(2t) + 4\sin^2(2t)}\, dt \rightarrow \frac{1}{2}$ area of eclipse $\frac{1}{2}$

~~$= 20 \int_0^{\pi} \frac{1}{21\cos^2(2t) + 4}\, dt$~~

$= \left(\frac{10}{\text{area of eclipse}} \right)$

(c) Is the vortex field $\boldsymbol{F}$ conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.

No. The domain is not simply connected.

2. (10 points) You do not need to simplify your answers.

(a) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where C is the straight line from $P = (1,2,3)$ to $Q = (4,5,6)$ and

$$\boldsymbol{F} = \left\langle \frac{2xy}{x^2+z}, \ln(x^2+z), \frac{y}{x^2+z} \right\rangle.$$

Box your answer

$r(t) = \langle 1+t, 2+t, 3+t \rangle, \ 0 \le t \le 3$

F is conservative: $\nabla f = F$ where $f(x,y,z) = y\ln(x^2+z)$

hence, $\int_C F \cdot dr = f(Q) - f(P)$

$= \boxed{5\ln(22) - 2\ln(4)}$

(b) Suppose that $\boldsymbol{F} = \langle y, z, 0 \rangle$ and that S is the surface parametrized by

$$G(u,v) = (u^3 - v, u + v, v^2), \qquad 0 \le u \le 2, \quad 0 \le v \le 3.$$

with downward-pointing normal.
Fill in the limits and integrand of the integral below so that it equals $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

$G(u,v) = \langle u^3 - v, u+v, v^2 \rangle$

$T_u = \langle 3u^2, 1, 0 \rangle$

$T_v = \langle -1, 1, 2v \rangle$

$N(u,v) = \begin{vmatrix} i & j & k \\ 3u^2 & 1 & 0 \\ -1 & 1 & 2v \end{vmatrix} = \langle 2v, -6u^2 v, 3u^2+1 \rangle$

$\iint_S F \cdot dS = \int_0^3 \int_0^2 F(G(u,v)) \cdot N(u,v)\, du\, dv = \int_0^3 \int_0^2 \langle u+v, v^2, 0 \rangle \cdot \langle 2v, -6u^2 v, 3u^2+1 \rangle \, du\, dv$

$= \int_0^3 \int_0^2 2uv + 2v^2 - 6u^2 v^3 \, du\, dv$

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_0^3 \int_0^2 2uv + 2v^2 - 6u^2 v^3 \quad du\, dv$$

You do not need to show work on this page.

3. (10 points) (a) Give a parametrization $G : D \to S$, where S is the triangle in $\mathbb{R}^3$ with vertices $(2,0,0)$, $(0,1,0)$, $(0,0,1)$ in the plane $\frac{1}{2}x + y + z = 1$. Be sure to explicitly specify the domain D and call your parameters u and v.

$G(u,v) = \langle u, v, 1 - \frac{1}{2}u - v \rangle$

$0 \le u \le 2, \; 0 \le v \le 1 - \frac{1}{2}u$ $\quad z = 1 - \frac{1}{2}x$

$y = 1 - \frac{1}{2}x$

$D = \{ \; 0 \le u \le 2 \;, \; 0 \le v \le 1 - \frac{1}{2}u \; \}$

$G(u,v) = \langle \; u \;, \; v \;, \; 1 - \frac{1}{2}u - v \; \rangle$

(b) Let S be the portion of the cylinder $x^2 + y^2 = 1$ between the xy-plane and the plane $x + z = 2$. We parametrize S by $G(u,v) = \langle \cos u, \sin u, v \rangle$, where (fill these in)

$0 \le u \le 2\pi \;, \; 0 \le v \le \cancel{2 - \sqrt{1-\sin^2 u}}$ $\quad 2 - \cos u$

$y^2 + (2-z)^2 = 1$

$\cancel{y^2 + 4 - 4z + z^2 = 1}$

$2 - z = \pm\sqrt{1-y^2}$

$z = 2 \pm \sqrt{1-y^2} = 2 - \sqrt{1-\sin^2 u} = 2 - \sqrt{\cos^2 u} = 2 - \cos u$

The surface integral $\iint_S x \, dS$ is (circle one) negative zero positive

$\cancel{\iint_S \cos u} \, dS$

Circle the correct answers.

4. (3 points) Let $\boldsymbol{F}$ be a vector field on an open connected domain $\mathcal{D}$ with continuous second order partial derivatives. Which of the following statements are always true, and which are only true if $\mathcal{D}$ is simply connected?

   (a) If $\operatorname{curl}(\boldsymbol{F}) = 0$ then $\boldsymbol{F}$ is conservative.

   Always true   Only true if $\mathcal{D}$ is simply connected

   (b) If $\boldsymbol{F}$ has a potential function then $\boldsymbol{F}$ is conservative.

   Always true   Only true if $\mathcal{D}$ is simply connected

   (c) If $\boldsymbol{F}$ is conservative then $\operatorname{curl}(\boldsymbol{F})$ is zero.

   Always true   Only true if $\mathcal{D}$ is simply connected

5. (5 points) Which of the following statements is true for all vector fields, and which is true only for conservative vector fields?

   (a) The line integral along a path from P to Q does not depend on which path is chosen.

   True for all vector fields   Only true for conservative vector fields

   (b) The line integral around a closed curve is zero.

   True for all vector fields   Only true for conservative vector fields

   (c) The line integral over an oriented curve $\mathcal{C}$ does not depend on how $\mathcal{C}$ is parametrized as long as each parametrization preserves the orientation of $\mathcal{C}$.

   True for all vector fields   Only true for conservative vector fields

   (d) The (vector) line integral is equal to the (scalar) line integral of the tangential component along the curve.

   True for all vector fields   Only true for conservative vector fields

   (e) The line integral changes sign if the orientation of the curve is reversed.

   True for all vector fields   Only true for conservative vector fields

xz x

6. (12 points) Multiple choice. Circle the correct answer.

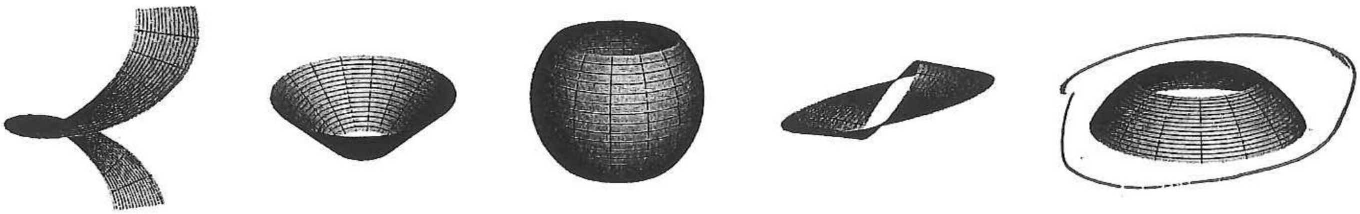
(a) Consider the vector field $\boldsymbol{F} = \langle xz, e^z - yz, \cos x\rangle$. Is there a function f such that $\boldsymbol{F} = \nabla f$?

f_{xz} f_{zx}

Yes   No   We don't have enough information

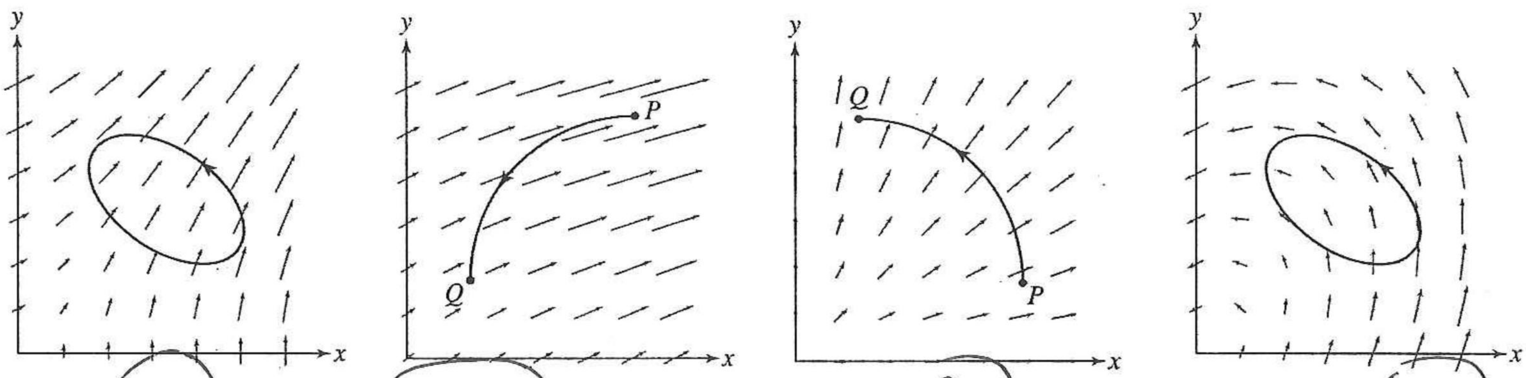
$f_{xz} \neq f_{zx}$

(b) $G(\theta, \phi) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$ with $0 \leq \theta \leq 2\pi$ and $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$ parametrizes which of the surfaces below? (circle one)



(c) Consider the line integrals $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for the vector fields $\boldsymbol{F}$ and paths $\boldsymbol{r}$ below.
Exactly **two** of the line integrals are zero, **one** is positive, and the remaining **one** is negative.
Circle "negative", "zero", or "positive" below each picture to indicate your answers.

Note: the closed curves are oriented counterclockwise and the others are oriented $P \to Q$.



negative zero positive   negative zero positive   negative zero positive   negative zero positive

(d) $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ is zero if (circle one)

$\boldsymbol{F}$ is tangent to S at every point $\boldsymbol{F}$ is perpendicular to S at every point