# Math 32B-1: Calculus of Several Variables

Instructor: Alpár R. Mészáros

First Midterm, January 25, 2016

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Discussion section (choose one): 1A 1B 1C 1D 1E 1F

**Rules:**

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. I will withhold your papers for **two** weeks after grading, after this period you can have them back.
- No calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the proctors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear statements. You may lose points in the lack of justification of your answers. In particular, there will be **no credit given only for the final answer**, in the lack of the computations that lead there.
- Theorems and formulas from the lectures and homework assignments may be used in order to justify your solution. In this case state clearly the theorem/formula you are using.
- This exam has **3** problems and is worth **25 points**.
- The problems are **not necessarily ordered** with respect to difficulty!
- I wish you success!

| Problem | Score |
|---|---|
| Exercise 1 | 5 |
| Exercise 2 | 9 |
| Exercise 3 | 4 |
| Total | 18 |

$\boxed{5}$

**Exercise 1** (5+6 = 11 points).

Let $D$ stand for the two dimensional domain which represents the circular sector of the unit circle centered at the origin, between the angles $\pi/4$ and $3\pi/4$ (in the trigonometric direction). This has a boundary which is composed by 3 curves: a line segment between the points $(0,0)$ and $(\sqrt{2}/2, \sqrt{2}/2)$, a piece from the unit circle, and a line segment between the points $(-\sqrt{2}/2, \sqrt{2}/2)$ and $(0,0)$.

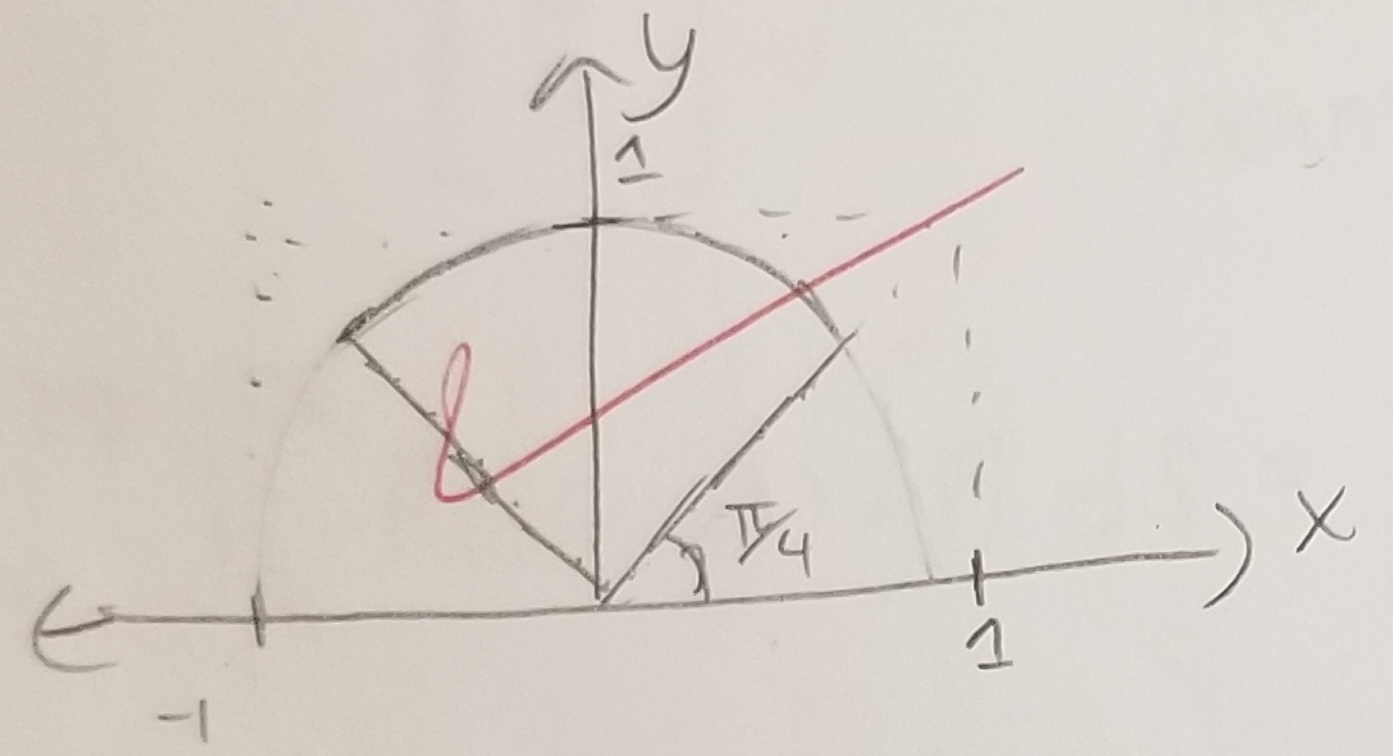
**Part 1**

(1)-1p Sketch the domain $D$.

(2)-2p Compute the area of $D$ using integration. *Hint:* you might use any integration technique, including Green's theorem.

(3)-2p Suppose that $\partial D$ has counterclockwise orientation. Compute $\oint_{\partial D} J \cdot dr$, where $J(x,y) := (x^2, y^2)$ and show that this integral does not depend on the parametrization of $\partial D$. Justify your answers!

1.)




2.) $G(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$ $r \in [0,1]$, $\theta \in [\pi/4, 3\pi/4]$

$\| Jac(G(r,\theta)) \| = r$

$\int_0^1 \int_{\pi/4}^{3\pi/4} r\,dr\,d\theta = \frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^1 = \pi/4$

Check: area(circle) $= \pi r^2$,

$1/4\ \pi(1)^2 = \pi/4$

3.) $J(x) = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$, curl$(J) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & 0 \end{vmatrix} = \hat{k}(0-0)$

$\left.\begin{array}{l} \int J_1\,dx = \frac{x^3}{3} + g(y) \\ \int J_2\,dy = y^3/3 + h(x) \end{array}\right\}$ $j(x,y) = \frac{x^3}{3} + \frac{y^3}{3}$

J is a conservative field defined on the entire domain, so any closed path parameterized in any way will have a $\oint_C J \cdot dr = 0$

**Part 2**

Now, we look the previously defined domain $D$ in $\mathbb{R}^3$, and translate it along the $z$-axis in a parallel way with the $xy$-axis ('lift it up', to arrive to height 5) to obtain the surface $S := \{(x, y, 5) \in \mathbb{R}^3 : (x, y) \in D\}$.

(4)-2p Give a parametrization $G : D \to S$ of the surface $S$, using $D$ as the base domain.

(5)-2p Compute the normal vectors at any point of $S$, using the previous parametrization. We are interested in the normals which are pointing "downwards."

(6)-2p We suppose that raindrops are falling with given a velocity $F(x, y, z) = (0, 0, -z)$. Compute the flux of the rain on the surface $S$, if we use the orientation given by the normal computed in (5).

4.) $G(x,y) = \begin{bmatrix} x \\ y \\ 5 \end{bmatrix}$ ✓

5.) The normal vector is orthogonal to the $z=5$ plane, so all unit normal vectors pointing down are $\vec{n} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ ✓

Use parametrization -.5pt

6.) $\int_{\cancel{\pi/4}}^{\cancel{3\pi/4}} \int_{\cancel{0}}^{\cancel{r}} \vec{F}(G(x,y)) \cdot \hat{\vec{n}}\, dS = \int_{\cancel{\pi/4}}^{\cancel{3\pi/4}} \int_{\cancel{0}}^{r} (0 + 0 + (-5)(-1))\, dS$

$= 5\,\text{area}(S) = \frac{5\pi}{4}$

-.5pt

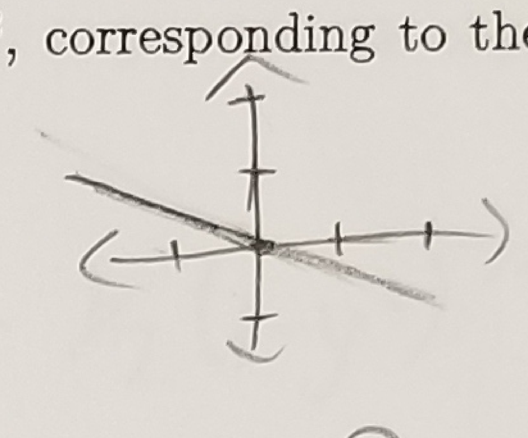
**Exercise 2** (7 points).

Let us consider the two dimensional vector field $F(x,y) = \left(\frac{1}{2\sqrt{x+2y}}, \frac{1}{\sqrt{x+2y}}\right)$.

0.5p (1)-1p Determine the domain of definition of the vector field $F$.

1p (2)-2p Is the vector field $F$ conservative on its domain of definition? If yes, determine its potential function. Justify your answer.

2p (3)-2p Let $D$ stand for a unit square defined as $D := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le 1\}$. Let $C = \partial D$ stand for the boundary of this square oriented counterclockwise. Give a parametrization of $C$. This parametrization needs to be 4 different curves $r_1, r_2, r_3, r_4 : [0,1] \to \mathbb{R}^2$, corresponding to the 4 sides of the square which all use 1 time unit to describe these pieces.



0.5p (4)-2p Compute $\oint_C F \cdot dr$, using the parametrization of $C$ from (3).

1.) F exists on the entire domain except on $x+2y=0$

$x+2y \le 0$

2.) $\int F(x,y)\,dx = \int \frac{1}{2\sqrt{x+2y}}\,dx = \int \frac{1}{2\sqrt{u}}\,du = \int \frac{1}{2}\,\frac{u^{1/2}}{1/2} = \sqrt{x+2y}$

$\int F(x,y)\,dy = \int \frac{1}{\sqrt{x+2y}}\,dx = \int \frac{1}{2\sqrt{u}}\,du = \frac{1}{2}\,\frac{u^{1/2}}{1/2} = \sqrt{x+2y}$ ✓

More detailes!

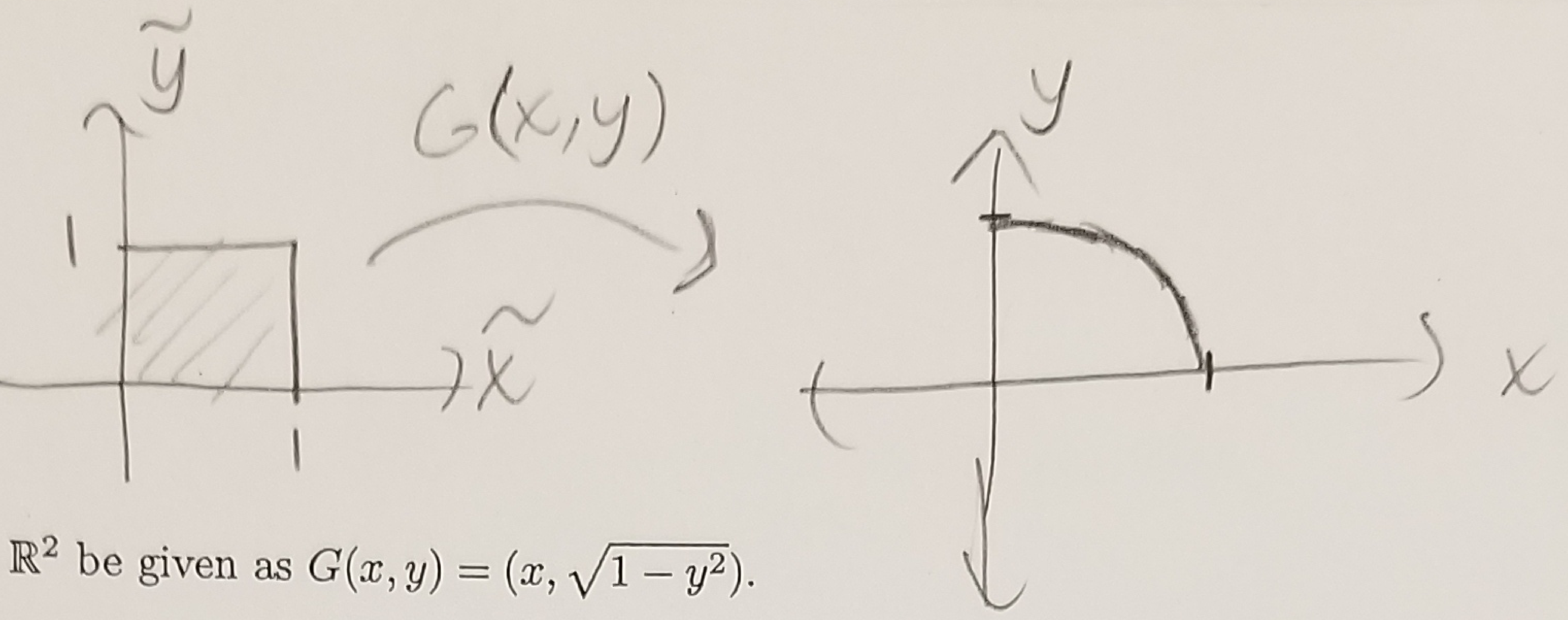
3.) $r_1(t) = \begin{bmatrix} 0+t \\ 0 \end{bmatrix}$ $r_2(t) = \begin{bmatrix} 1 \\ 0+t \end{bmatrix}$ ✓ $r_3(t) = \begin{bmatrix} 1-t \\ 1 \end{bmatrix}$ ✓ $r_4(t) = \begin{bmatrix} 0 \\ 1-t \end{bmatrix}$ ✓

4.) Since F has a potential function, and $\lim_{w \to 0} \int_w^1 \vec{F}(\vec{r}(t)).\vec{T}\,ds$ ? why?

exist, since the curve is closed,

More detailes!

$\oint_C \vec{F} \cdot d\vec{r} = 0$




**Exercise 3** (7 points).

Let $G : \mathbb{R} \times [0,1] \to \mathbb{R}^2$ be given as $G(x,y) = (x, \sqrt{1-y^2})$.

(1)-1p Compute $\operatorname{div}(G)$. Determine the points $(x,y)$, where $G(x,y)$ is incompressible. *Hint:* incompressibility is related to the sign of the divergence.

(2)-2p Compute the Jacobian matrix and Jacobian determinant of $G$.

(3)-2p Let $D$ be the unit square defined as $D := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le 1\}$ and let $\tilde{D} := G(D)$, i.e. the image of $D$ through $G$. Show that $\operatorname{area}(\tilde{D}) = 1$.

(4)-2p What does the domain $\tilde{D}$ represent geometrically? Describe as precisely as possible this object. Justify your answer!

1.) $\operatorname{Div}(G) = \nabla \cdot G = \frac{\partial x}{\partial x} + \frac{\partial}{\partial y}\left(\sqrt{1-y^2}\right) = 1 + \frac{1}{2}(1-y^2)^{-1/2}(-2y)$

$= 1 + \frac{-2y}{2\sqrt{1-y^2}}$

0.5 $G(x,y)$ is incompressible where $\cancel{1-y^2 \le 0} \Rightarrow y \ge 1$

2.) $x = r\cos\theta$, $y = r\sin\theta$

$\cancel{\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix} = r}$

0

3.) $\cancel{G(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}}$ $\quad \theta = \tan^{-1}(y/x) \quad \theta \in [0, \frac{\pi}{2}]$

0 $\int$

4.) $\tilde{D}$ is the quarter unit circle in the first quadrant.

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