Math 32B-1: Calculus of Several Variables

Instructor: Alpár R. Mészáros

First Midterm, January 25, 2016

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Discussion section (choose one): 1A 1B 1C 1D 1E 1F

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. I will withhold your papers for **two** weeks after grading, after this period you can have them back.
- No calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the proctors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear statements. You may lose points in the lack of justification of your answers. In particular, there will be **no credit given only for the final answer**, in the lack of the computations that lead there.
- Theorems and formulas from the lectures and homework assignments may be used in order to justify your solution. In this case state clearly the theorem/formula you are using.
- This exam has **3** problems and is worth **25 points**.
- The problems are **not necessarily ordered** with respect to difficulty!
- I wish you success!

Problem	Score
Exercise 1	5
Exercise 2	9
Exercise 3	4
Total	18

Exercise 1 (9 points).

Let D be the following two dimensional domain: $D := \{(x, y) \in \mathbb{R}^2 : \ 0 \le x \le \pi/2; -\sin(x) \le y \le \sin(x)\}$.

Part 1

(1)-2p Sketch the domain D and compute its area. Justify your answer! 1

(2)-2p Let $f : D \to \mathbb{R}$ be defined as $f(x,y) = xy^{2019}$. Compute $\iint_D f(x,y)\,\mathrm{d}A$. Justify your answer! 2

Part 2

(3)-1p Let $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$ denote the inverse of sin, i.e. $\arcsin(\sin(x)) = x$ for $x \in [-\pi/2, \pi/2]$ and $\sin(\arcsin(y)) = y$ for $y \in [-1,1]$. Show that 1

$$\cos^2(\arcsin(x)) = 1 - x^2, \text{ for } x \in [-1,1].$$

(4)-2p From (3), deduce with justification that 1

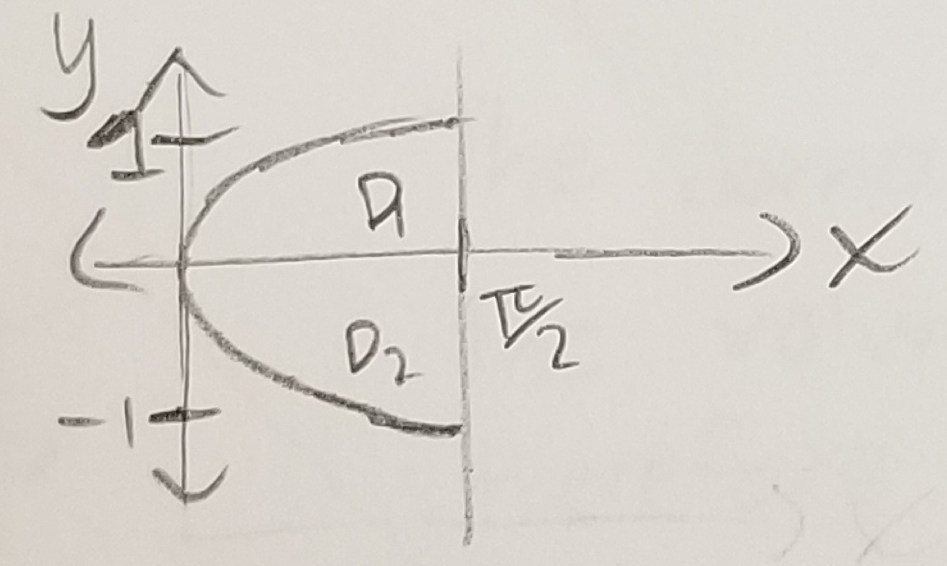
$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \text{ for } x \in (-1,1).$$

(5)-2p Compute $\iint_D \frac{x^2}{\sqrt{1-y^2}}\,\mathrm{d}A$. 0

1.) $\sin(0) = 0$

$\sin(\frac{\pi}{2}) = 1$

$-\sin(\frac{\pi}{2}) = -1$



Area?

2.) $\int_{x=0}^{\frac{\pi}{2}} \int_{-\sin(x)}^{\sin(x)} xy^{2019}\,dy\,dx = \int_0^{\frac{\pi}{2}} x\left[\frac{y^{2020}}{2020}\right]_{-\sin(x)}^{\sin(x)} dx = \int_0^{\frac{\pi}{2}} x\,\frac{\sin(x)^{2020} + \sin(x)^{2020}}{2020}$

$= \int_0^{\frac{\pi}{2}} \frac{2x\sin^{2020}(x)}{2020}\,dx$

$f(x,y) = xy^{2019}$

$\int_{y=-1}^{1} \int_{x=\sin^{-1}(y)}^{\frac{\pi}{2}} xy^{2019}$ $\quad f(x,-y) = -xy^{2019}$

Because $f(x,y) = -f(x,-y)$, and the domain is symmetrical ~~acro~~ across the x-axis

$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$

$= \iint_{D_1} f(x,y)\,dA - \iint_{D_1} f(x,y)\,dA = 0$

Exercise 2 (9 points).

Let W, U be the following three dimensional domains: $W := \{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2 \le z \le 4\}$ and $U := \{(x,y,z) : x^2+y^2 \le 4 \text{ and } 0 \le z \le x^2+y^2\}$.

(1)-3p Sketch the domain W and compute its volume. Justify your answer.

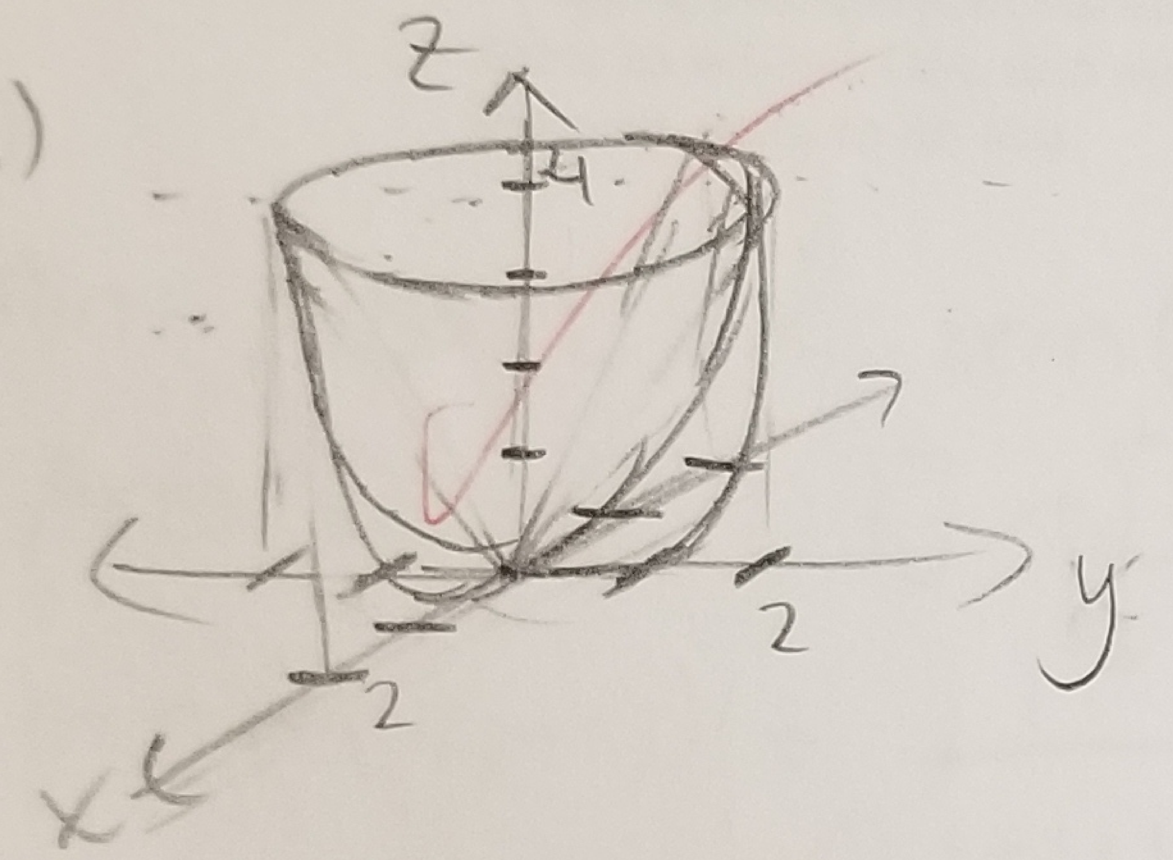
(2)-3p Sketch the domain U and compute its volume. Justify your answer. *Hint:* you might use (1) and you can use the volume of the right cylinder without a proof!

(3)-3p Compute $\iiint_W yz\,dV + \iiint_U yz\,dV$. Justify your answer! *Hint:* you might use the additive property of the integral with respect to decomposition of domains.

$z = r^2 = x^2 + y^2$

$\sqrt{z} = r = \sqrt{x^2+y^2}$

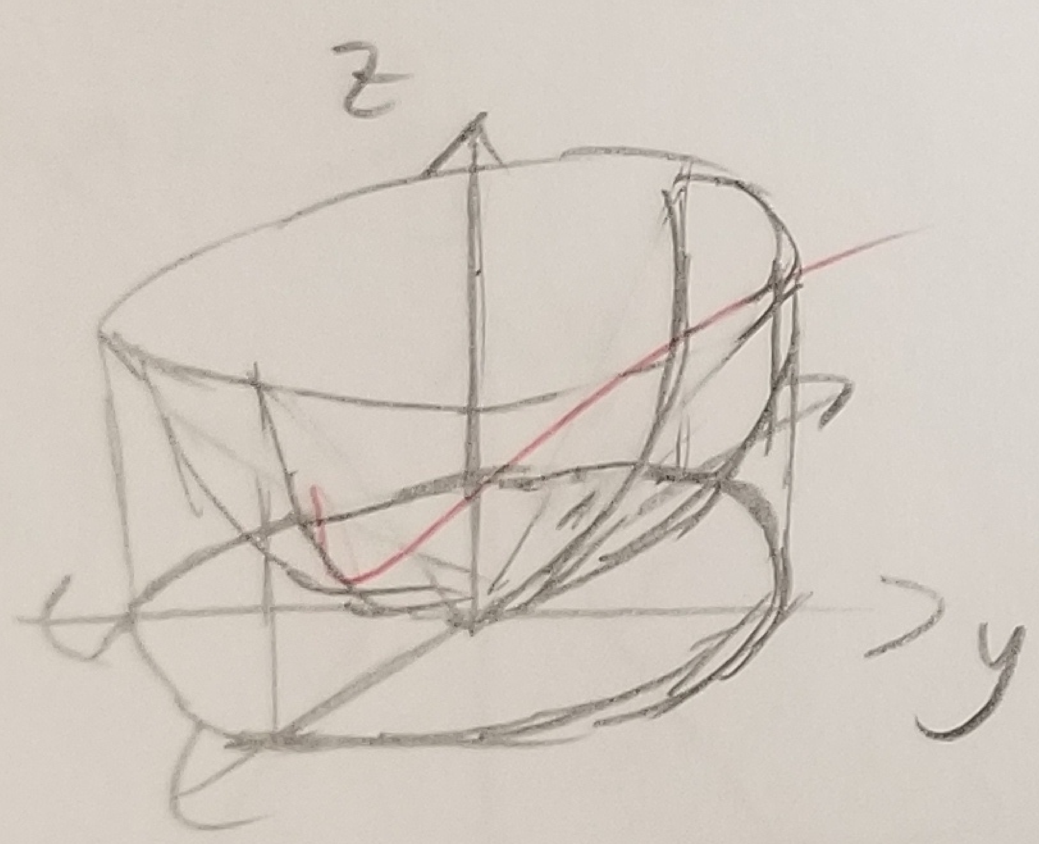
1.)



$\int_{z=0}^{4}$ area(circle w/ radius $\sqrt{z}$) dz

$= \int_0^4 \pi z \, dz = \left[\pi \frac{z^2}{2}\right]_0^4 = 8\pi$

2.)



The domain of 2 is a cylinder with the volume of domain 1 missing therefore

$\iiint_U dV = \text{Vol(cylinder)} - \iiint_W dV$

$= 4\pi 2^2 - 8\pi = 8\pi$

3.) $\iiint_W yz\,dV + \iiint_U yz\,dV = \iiint_{\text{cylinder}} yz\,dV \qquad y = r\sin\theta$

$\int_{z=0}^{4}\int_{\theta=0}^{2\pi}\int_{r=0}^{2} zr^2 \sin\theta \, dr\, d\theta\, dz = \int_0^4\int_0^{2\pi} \frac{8z}{3}\sin\theta\, d\theta\, dz$

$= \int_0^4 \frac{8z}{3}\left[\cos\theta\right]_0^{2\pi} dz = \int_0^4 \frac{8z}{3}(1-1)\,dz = 0$

Exercise 3 (7 points).

Part 1 - 3p
Let D be the two dimensional square represented by $D := \{(x, y) \in \mathbb{R}^d : -1 \leq x \leq 1;\ -1 \leq y \leq 1\}$.
Show that

$$\frac{4}{3} \leq \iint_D \frac{1}{1 + x^8 + y^6}\, dA \leq 4.$$

Hint: it is not necessary to compute precisely the integral. Compute instead the minimum and the maximum of the integrand on D.

Part 2 - 4p

Using two dimensional integration, compute the area of the equilateral triangle with side length $r > 0$.
Hint: to achieve this, first describe analytically such a triangle (via x and y coordinates). There are many different ways to do this.

1.)

Min = Area(D) × min(f(x,y))

Area(D) = $[-1,1] \times [-1,1] = 4$

$\min(f(-1,-1)) = \frac{1}{1+1+1} = \frac{1}{3}$ ✓

$\cancel{\text{Min}} = \frac{4}{3}$

Max = Area(D) × max(f(x,y))

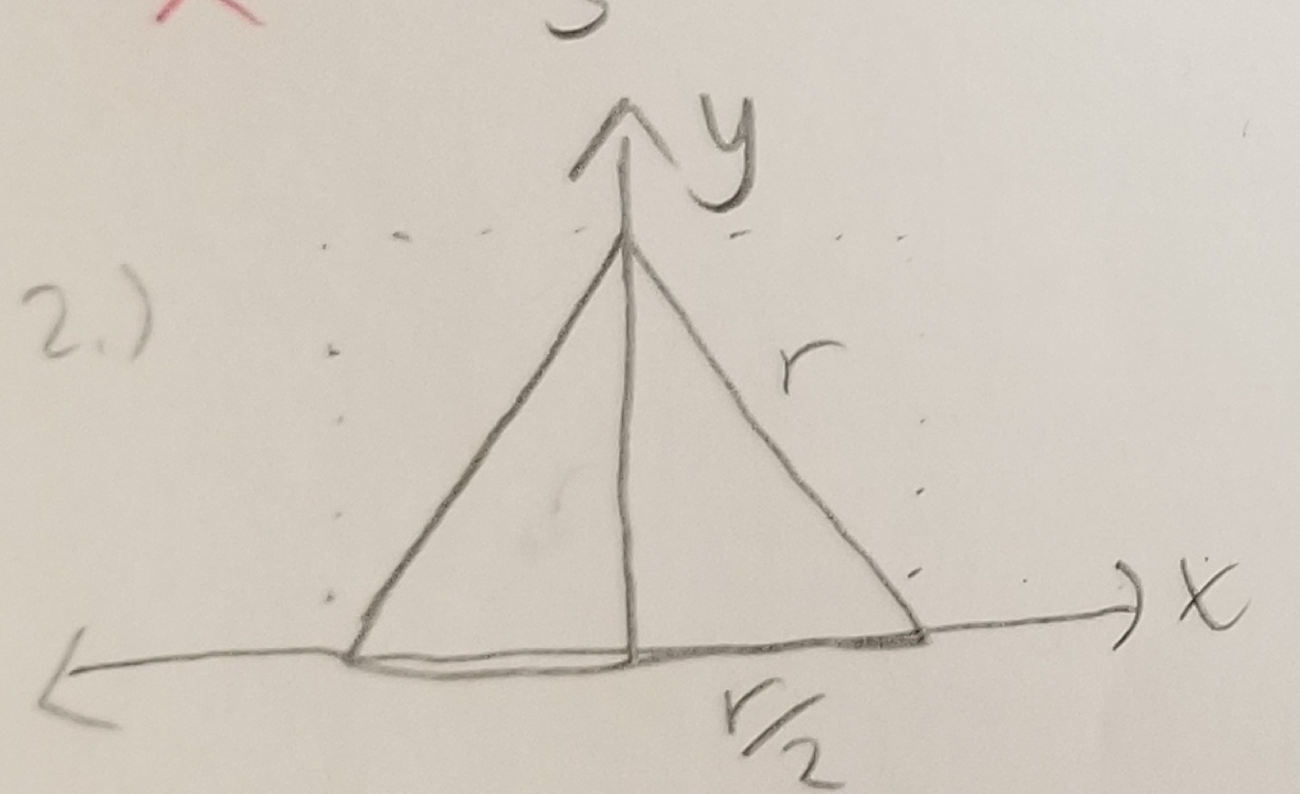
Area(D) = 4 ✓

$\max(f(0,0)) = \frac{1}{1} = 1$ ✓

$\text{Ma}\cancel{\text{x}} = 4(1) = 4$

3/3

2.)



$y = r - 2x$

1/4

$$\text{Area} = 2\int_{x=0}^{\cancel{r/2}} \int_{y=0}^{\cancel{r-2x}} dy\, dx = \int_0^{r/2} r - 2x\, dx = \left[rx - x^2\right]_0^{r/2}$$

$$= 2\left(\frac{r^2}{2} - \frac{r^2}{4}\right) = \frac{2r^2}{4} = \cancel{\frac{r^2}{2}}$$