Exercise 1 (11+15=26 points).

This problem has **2 parts**, the second part is depending on the first one. Please **turn the page**, when you finished with the first one.

Part 1 (11 points)

Let S be the surface which is a quarter of the unit sphere in $\mathbb{R}^3$ centered at the origin, i.e.

$$S := \{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2+z^2=1, \ y \geq 0, z \geq 0\}.$$

Let moreover D_1 stand for the half disk in the xOy plane given by $D_1 := \{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 \leq 1, \ y \geq 0\}$ and let D_2 stand for the half disk in the xOz pane given by $D_2 := \{(x,z) \in \mathbb{R}^2 : x^2+z^2 \leq 1, \ z \geq 0\}$.

(1)-1.5p Sketch the surface S and the disks D_1 and D_2 in a single figure in 3D.

(2)-1p Give a parametrization of S using D_1 as base domain, i.e. we are looking for a map $G : D_1 \to \mathbb{R}^3$ such that $S = G(D_1)$. *Hint:* notice that S can be parametrized as a graph. Do not use spherical coordinates!

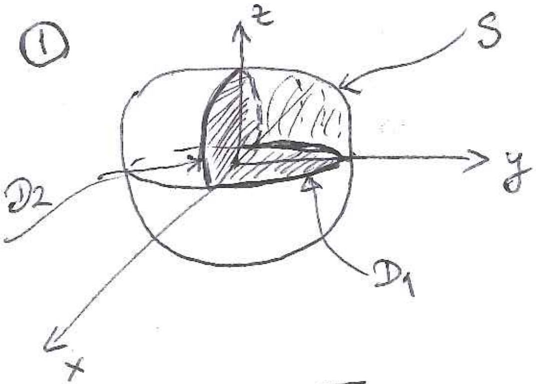
(3)-2.5p Compute the outward normal vectors to S (the ones that are **not** pointing towards the origin), using the parametrization from (2). Determine the points on S, where the parametrization is *not regular*. *Hint:* you need to check whether the normal is pointing outwards and where is it vanishing.

(4)-2p Using integration, compute the surface area of S. *Hint:* you can use the normal computed in (3), then use polar coordinates.

(5)-2p Looking at D_1 and D_2 as surfaces, give their (trivial) parametrizations (using base domains D_1 and D_2, respectively), such that the associated normal vectors are outwards (pointing to the negative z-axis and to the negative y-axis, respectively). Compute these normal vectors.

(6)-2p Using integration, compute the volume of the 3D solid object bounded by S, D_1 and D_2.

(1)



(2) $z = \sqrt{1-x^2-y^2}$, therefore

$G: D_1 \to \mathbb{R}^3$; $G(x,y) = (x;y;\sqrt{1-x^2-y^2})$

(3) $T_x = \left(1;\ 0;\ -\frac{x}{\sqrt{1-x^2-y^2}}\right)$ if $x^2+y^2 \neq 1$

$T_y = \left(0;\ 1;\ -\frac{y}{\sqrt{1-x^2-y^2}}\right)$

$\Rightarrow$ normal: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{x}{\sqrt{1-x^2-y^2}} \\ 0 & 1 & -\frac{y}{\sqrt{1-x^2-y^2}} \end{vmatrix}$

$= \left(\frac{x}{\sqrt{1-x^2-y^2}};\ \frac{y}{\sqrt{1-x^2-y^2}};\ 1\right)$

Plugging in $x=0=y \Rightarrow N=(0,0,1)$, so it is clearly pointing outwards

Non-regular points: we are not able to do the computation of the normal where $x^2+y^2=1$, so these are non-regular.

(4) $\text{area}(S) = \iint_S 1\, dS = \iint_{D_1} \|N\|\, dA =$

$= \iint_{D_1} \sqrt{1+\frac{x^2+y^2}{1-x^2+y^2}}\, dA = \iint_{D_1} \frac{1}{\sqrt{1-x^2-y^2}}\, dA$

$= \int_0^{\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\, dr\, d\theta = \pi\left[-\sqrt{1-r^2}\right]_0^1 = \pi.$

(5) Parametrization of D_1:

$G_1: D_1 \to \mathbb{R}^3$; $G_1(x,y) = (x,y;0)$

Normal vector $N_{D_1}: (0,0,-1)$.

Parametrization of D_2:

$G_2: D_2 \to \mathbb{R}^3$; $G_2(x,z) =$

$= (x,0;z)$

Normal vector: $N_{D_2}: (0,-1;0)$.

(6.) volume (solid object) $= \iiint_W 1\,dV = \iint_{D_1} \sqrt{1-x^2-y^2}\, dA(x,y) =$

$$= \int_0^{\pi}\int_0^{1} r\sqrt{1-r^2}\, dr\, d\theta = \pi \left[-\frac{2}{2\cdot 3}(1-r^2)^{\frac{3}{2}} \right]_0^1 = \frac{\pi}{3}.$$

polar coord

Exercise 1 - Part 2 (15 points)

(1)-2p Give a parametrization of ∂S, the boundary of the surface S, which follows the natural orientation, in correspondence with the normal vectors from Part 1/(3). *Hint:* these are two half circles.

(2)-2p Let us consider the constant vector field $F(x,y,z) = (5,6,7)$. Find a vector potential of F, i.e. a vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F = \mathrm{curl}(A)$. Justify your answer!

(3)-3p Compute $\iint_S F \cdot \mathrm{d}S$ and $\oint_{\partial S} A \cdot \mathrm{d}r$.

(4)-2p Compute $\iint_{D_1} F \cdot \mathrm{d}S$, using the orientation from Part 1/(5).

(5)-1p Use the divergence theorem and (3) and (4) to compute $\iint_{D_2} F \cdot \mathrm{d}S$, when using the orientation from Part 1/(5).

Let $G(x,y,z) = (x^2+y^2, 0, z^2)$.

(6)-1p Compute the flux of G on D_1, using the orientation from Part 1/(5).

(7)-1p Compute the flux of G on D_2, using the orientation from Part 1/(5).

(8)-3p Using the divergence theorem, compute the flux of G on S, using the orientation from Part 1/(3).

(1.) ∂S has two parts: (i) part of ∂D_1 : $r_1 : [0,\pi] \to \mathbb{R}^3$; $r_1(t) = (\cos t; \sin t; 0)$.
(ii) part of ∂D_2 : $r_2 : [0,\pi] \to \mathbb{R}^3$; $r_2(t) = (-\cos t; 0; \sin t)$.

(2.) $A = (A_1, A_2, A_3)$

$$\mathrm{curl}(A) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ A_1 & A_2 & A_3 \end{vmatrix} = (\underbrace{\partial_y A_3 - \partial_z A_2}_{=5}; \underbrace{\partial_z A_1 - \partial_x A_3}_{=6}; \underbrace{\partial_x A_2 - \partial_y A_1}_{=7}).$$

Clearly, by setting ~~A(x,y,z)~~ $A(x,y,z) = (6z; 7x; 5y)$, we have the desired vector potential. Another different one would be $A = (6z; 7x-5z; 0)$

(3.) $$\iint_S F \cdot dS = \iint_S \mathrm{curl}(A) \cdot dS \underset{\text{Stokes}}{=} \oint_{\partial S} A \cdot dr$$ (since both F & A are smooth and defined on the whole $\mathbb{R}^3$).

Therefore, it is enough to compute only one of the integrals.

For instance $$\oint_{\partial S} A \cdot dr = \int_0^\pi (0; 7\cos t; 5\sin t) \cdot (-\sin t; \cos t; 0)\, dt + \int_0^\pi (6\sin t; -7\cos t; 0) \cdot (\sin t; 0; \cos t)\, dt$$

$$= \int_0^\pi 7\cos^2 t + 6\sin^2 t \, dt = \int_0^\pi \cos^2 t + 6 \, dt = 6\pi + \int_0^\pi \cos^2 t \, dt = 6\pi + \int_0^\pi \frac{1+\cos(2t)}{2} dt$$

$$= 6\pi + \frac{\pi}{2} + \frac{1}{2}\left[\frac{\sin(2t)}{2}\right]_0^\pi = \frac{13\pi}{2}.$$

④ $\iint_{D_1} F\cdot dS = \iint_{D_1} (5,6,7)\cdot(0,0,-1)\,dA = -7\,\text{area}(D_1) = -\frac{7\pi}{2}.$

⑤ $$\underbrace{\iint_{D_1} F\cdot dS}_{=-\frac{7\pi}{2}} + \iint_{D_2} F\cdot dS + \underbrace{\iint_{S} F\cdot dS}_{\frac{13\pi}{2}} = \iiint_{W} \operatorname{div}(F)\,dV = 0.$$

W solid obj

$$\Rightarrow \iint_{D_2} F\cdot dS = -3\pi$$

⑥ $\iint_{D_1} G\cdot dS = \iint_{D_1} (G\cdot N_1)\,dA = \iint_{D_1} (x^2+y^2;0;0)\cdot(0,0,-1)\,dA = 0.$

⑦ $\iint_{D_2} G\cdot dS = \iint_{D_2} (G\cdot N_2)\,dA = \iint_{D_2} (x^2;0;z^2)\cdot(0;-1;0)\,dA = 0.$

⑧ $$\iint_{S} G\cdot dS = \iiint_W \operatorname{div}(G)\,dV - \underbrace{\iint_{D_1} G\cdot dS}_{=0\ (6)} - \underbrace{\iint_{D_2} G\cdot dS}_{=0\ (7)} = \iiint_W \operatorname{div}(G)\,dV =$$

$$= \iiint_W (2x+2z)\,dV = 2\iint_{D_1}\int_0^{\sqrt{1-x^2-y^2}} (x+z)\,dz\,dA(x,y) =$$

$$= \iint_{D_1} 2x\sqrt{1-x^2-y^2}\,dA(x,y) + \iint_{D_1} 1-(x^2+y^2)\,dA(x,y)$$

$$= \int_0^{\pi}\int_0^1 2r\cos\theta\, r\sqrt{1-r^2}\,dr\,d\theta + \int_0^{\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

polar coordinates

$$= 0 + \pi\left[-\frac{1}{4}(1-r^2)^2\right]_0^1 = \frac{\pi}{4}.$$

since $\int_0^{\pi}\cos\theta\,d\theta = 0$

Exercise 2 (12 points).

Let us consider the function $f(x,y) = \ln(\sqrt{x^2+y^2})$ on a subset of the plane, which we aim to find below.

(1)-1p Determine the domain of definition of f. Is it simply connected? Justify your answer!

(2)-1p Compute ∇f on the domain of f.

(3)-2p Compute Δf on the domain of f, where we use the notation $\Delta f := \operatorname{div}(\nabla f)$.

(4)-2p Show that $\operatorname{div}(f\nabla f) = \|\nabla f\|^2$, on the domain of f.

(5)-2p For $r > 0$, define $D_r := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le r^2\}$, and we orient its boundary ∂D_r counterclockwise. Compute $\oint_{\partial D_r} \nabla f \cdot dr$.

(6)-2p Consider the normals to ∂D_r that are pointing outwards (**not** towards the origin), and compute the flux of ∇f through ∂D_r.

(7)-2p We would like to give a sense to $\iint_{D_r} \Delta f \, dA$. Suppose that we have the 2D divergence theorem given as: for a vector field $F : D_r \to \mathbb{R}^2$ we have

$$\iint_{D_r} \operatorname{div}(F)\, dA = \oint_{\partial D_r} (F \cdot N)\, ds,$$

where N is the outward pointing unit normal vector to ∂D_r. Using this theorem, what should be the value of $\iint_{D_r} \Delta f \, dA$? Justify your answer!

(1.) Domain of definition $(x,y)\in\mathbb{R}^2 : x^2+y^2>0$; This is $\mathbb{R}^2\setminus\{(0,0)\}$ which is not simply connected.

(2.) $\partial_x f(x,y) = \frac{1}{\sqrt{x^2+y^2}}\partial_x\left(\sqrt{x^2+y^2}\right) = \frac{1}{\sqrt{x^2+y^2}}\cdot\frac{x}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$

Similarly $\partial_y f(x,y) = \frac{y}{x^2+y^2}$, so $\nabla f(x,y) = \left(\frac{x}{x^2+y^2} ; \frac{y}{x^2+y^2}\right)$
$(x,y)\neq(0,y)$

(3.) On $\mathbb{R}^2\setminus\{(0,0)\}$ we have $\partial_x\left(\frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2} + x(-1)\frac{1}{(x^2+y^2)^2}\cdot 2x = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$

Similarly $\partial_y\left(\frac{y}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$;

Therefore $\Delta f = \partial_x\left(\frac{x}{x^2+y^2}\right) + \partial_y\left(\frac{y}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2} = 0$.

(4.) We have on $\mathbb{R}^2\setminus\{(0,0)\}$: $\operatorname{div}(f\nabla f) = \partial_x(f\,\partial_x f) + \partial_y(f\,\partial_y f) =$
$= \partial_x f\cdot\partial_x f + f\,\partial^2_{xx} f + \partial_y f\,\partial_y f + f\,\partial^2_{yy} f = (\partial_x f)^2 + (\partial_y f)^2 + f\underbrace{\Delta f}_{=0 \text{ by (3)}} = \|\nabla f\|^2$.

(5.) Since $(0,0) \notin$ domain of f; and $(0,0) \in D$, we cannot use the fundamental theorem of line integrals in this case, so we compute $\oint_{\partial R} \nabla f \cdot dr$ "manually". Parametrization of ∂D_r:

$p(\theta): \begin{cases} r\cos\theta \\ r\sin\theta \end{cases}$; r is fixed and $\theta \in [0, 2\pi]$.

$$\oint_{\partial D_r} \nabla f \cdot dr = \int_0^{2\pi} \left(\frac{r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta} ; \frac{r\sin\theta}{r^2\sin^2 + r^2\cos^2\theta} \right) \cdot (-r\sin\theta ; r\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} -\frac{\cos\theta}{r} \cancel{r}\sin\theta + \frac{\sin\theta}{\cancel{r}} \cdot \cancel{r}\cos\theta\, d\theta = 0.$$

Also, geometrically one can see that $\nabla f \perp e'(\theta)$; therefore, the integral has to be 0.

(6.) $$\text{Flux} = \int_{\partial D_r} (\nabla f \cdot N)\, ds = \int_{\partial D_r} \left(\frac{x}{x^2+y^2} ; \frac{y}{x^2+y^2} \right) \cdot \frac{(x,y)}{r}\, ds = \int_{\partial D_r} \frac{1}{r} ds = \frac{2\pi r}{r} = 2\pi$$

Unit, $N = \frac{(x,y)}{r}$

or using the parametrization from (5.), we have

$$\int_{\partial D_r} (\nabla f \cdot N)\, ds = \int_0^{2\pi} \left(\frac{r\cos\theta}{r^2} ; \frac{r\sin\theta}{r^2} \right) \cdot (\cos\theta ; \sin\theta)\, r\, d\theta = 2\pi.$$

(7.) Notice that since we have $(0,0) \in D_r$, we cannot say that $\iint_{D_r} \Delta f\, dA = 0$; since $(0,0)$ is a singularity for Δf too.

Because of the 2D-div theorem, we might give a sense to this object by $\iint_{D_r} \Delta f\, dA = \int_{\partial D_r} (\nabla f \cdot N)\, ds = 2\pi$ (computed in (6))

N: outward unit normal

Exercise 3 (12 points).

Let us consider the potential function $\varphi(x,y) = x + \frac{x}{x^2+y^2}$ and the vector field $F(x,y) = \nabla\varphi(x,y)$.

(1)-2p Determine F together with its domain of definition.

(2)-2p Show that F is incompressible at any point of its domain. *Hint:* compute its divergence!

(3)-2p Show that F is irrotational at any point of its domain, i.e. $\mathrm{curl}_z F(x,y) = 0$.

(4)-2p Let $C := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$, centered at the origin. Show that F is tangential to C, except at the points $(\pm 1, 0)$.

(5)-1p Compute the flux of F through C.

(6)-2p Compute $\oint_C F \cdot dr$, where C is oriented clockwise.

(7)-1p Let R denote the circle with radius 1, centered at $(1,1)$, oriented counterclockwise. Compute $\oint_R F \cdot dr$.

(1) Domain of definition : $x^2+y^2 \neq 0 \Rightarrow \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\partial_x \varphi = 1 + \frac{1}{x^2+y^2} + x(-1)\frac{1}{(x^2+y^2)^2}2x = 1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} =: F_1$$

$$\partial_y \varphi = \frac{x}{(y^2+x^2)^2}(-1)2y = -\frac{2xy}{(x^2+y^2)^2} =: F_2$$

and $F = (F_1, F_2)$.

(2) We compute $\mathrm{div}(F) = 0$ on its domain of def.

$$\partial_x F_1 = \partial_x\left(1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right) = -\frac{2x}{(x^2+y^2)^2} - \frac{4x}{(x^2+y^2)^2} + \frac{4x^2}{(x^2+y^2)^3}\cdot 2x =$$

$$= -\frac{6x}{(x^2+y^2)^2} + \frac{8x^3}{(x^2+y^2)^3}$$

$$\partial_y F_2 = \partial_y\left(-\frac{2xy}{(x^2+y^2)^2}\right) = \frac{-2x}{(x^2+y^2)^2} + \frac{4xy}{(x^2+y^2)^3}\cdot 2y =$$

$$= \frac{-2x}{(x^2+y^2)^2} + \frac{8xy^2}{(x^2+y^2)^3}$$

So adding up, we find

$$\mathrm{div}(F) = -\frac{8x}{(x^2+y^2)^2} + \frac{8x(x^2+y^2)}{(x^2+y^2)^3} = 0.$$

5

(3) Compute $\partial_y F_1 = \partial_y \left(1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right) = -\frac{2y}{(x^2+y^2)^2} + \frac{4x^2 \cdot 2y}{(x^2+y^2)^3}$

and $\partial_x F_2 = \partial_x \left(-\frac{2xy}{(x^2+y^2)^2}\right) = -\frac{2y}{(x^2+y^2)^2} + \frac{4xy \cdot 2x}{(x^2+y^2)^3}$

Since $\partial_y F_1 = \partial_x F_2 \Rightarrow \mathrm{curl}_z(F) = 0$.

(4) On C; i.e. when $x^2+y^2=1$; we have

$$F = (F_1; F_2) = (1+1-2x^2; -2xy)$$

We will show that $F \perp N$; where the normal to C; $N=(x,y)$

Take $F \cdot N = (2-2x^2; -2xy)\cdot(x,y) = 2x - 2x^3 - 2xy^2 = 2x - 2x(x^2+y^2) = 0$,

So F is tangential to C.

Notice that on C, $F = (2-2x^2; -2xy) = 2(y^2; -xy) = 2y(y; -x)$

So $F=(0,0)$ at the points $(\pm 1; 0)$ on C, where it is not meaningful to say that F is tangential to C.

(5) Since F is tangential to C (by (4)), we have that

$$\int_C (F \cdot N)\, ds = \int_C 0\, ds = 0.$$

(6) $\oint_C F \cdot dr = \int_0^{2\pi} (2 - 2\cos^2\theta; -2\sin\theta\cos\theta) \cdot (\sin\theta; -\cos\theta)\, d\theta = 2\int_0^{2\pi} (\sin^3\theta + \sin\cos^2\theta)\, d\theta = 2\int_0^{2\pi} \sin\theta\, d\theta = 0$

$r = \begin{cases} x = \cos\theta \\ y = -\sin\theta \end{cases}$; $\theta \in [0, 2\pi]$

parametrization of C

---

(7) Since the only singularity of F is at $(0,0)$ and since $\|(1,1)\| = \sqrt{2}$, the disk centered at $(1,1)$ with radius 1, does not intersect $(0,0)$.

Since F is conservative, on a closed curve (not enclosing or containing singularities) its integral is 0.