MATH-32B | MIDTERM 2 | FALL 2017 | NAME: Steven Lara SECTION: 2E

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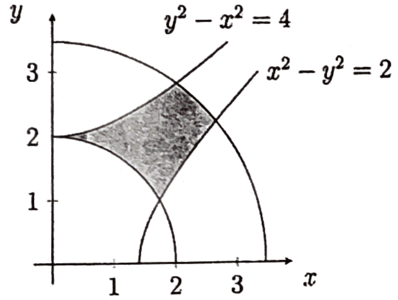
Instructions:

- This is a closed-book exam. Do not use any reference materials. No calculators.
- Do all work in this exam booklet. Organize your work in an unambiguous order. You may continue work to the backs of pages but if you do so indicate this.
- Show all necessary steps. Answers given without supporting work may receive 0 credit.

Problem #	Score
1	7
2	10
3	10
4	10
Total	37

Problem 1 [10pt].



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Let D be the region in the first quadrant bounded by the circles $x^2+y^2 = 4$ and $x^2+y^2 = 12$, and the hyperbolae $x^2-y^2 = 2$ and $y^2-x^2 = 4$. Use the change of variables $(x(u,v), y(u,v)) = (\sqrt{u+v}, \sqrt{u-v})$ to compute the integral

$$G(u,v) = (\sqrt{u+v}, \sqrt{u-v}) \quad \iint_D 4xy^3 dA.$$

$x^2+y^2=u$

$y^2-x^2=v$

$4 \le x^2+y^2 \le 12$

$-2 \le y^2-x^2 \le 4$

$$\text{Jac}(G) = \begin{vmatrix} \frac{1}{2\sqrt{u+v}} & \frac{1}{2\sqrt{u+v}} \\ \frac{1}{2\sqrt{u-v}} & -\frac{1}{2\sqrt{u-v}} \end{vmatrix}$$

$\frac{dx}{du}$ $\frac{dx}{dv}$

$\frac{dy}{du}$ $\frac{dy}{dv}$

$$\iint F(x(u,v), y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

absolute value

$$= \frac{-1}{4\sqrt{u+v}\cdot\sqrt{u-v}} - \frac{1}{4\sqrt{u+v}\cdot\sqrt{u-v}} = -\frac{1}{2\sqrt{u+v}\cdot\sqrt{u-v}}$$

$$= \int_{2}^{6}\int_{-2}^{4} \overset{2}{\cancel{4}}\cdot\cancel{\sqrt{u+v}}\cdot\sqrt{u-v}^{\cancel{3}\,2}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{u+v}}\cdot\cancel{\sqrt{u-v}}}\,dv\,du = \int_4^{12}\int_{-2}^{4} 2(u-v)\,dv\,du$$

$$= \int_4^{12}\int_{-2}^{4} 2u-2v\,dv\,du$$

$$= \int_4^{12}\left(2uv - v^2\Big|_{-2}^{4}\right)du = \int_4^{12}(8u-16+4u+4)\,du = \int_4^{12} 12u-12\,du = 6u^2-12u\Big|_4^{12}$$

$$= (6(144)-144) - (96-36)$$

$$= (5\cdot 144) - 60$$

$$= 720 - 60 = \boxed{660}$$

Problem 2 [10pt]. Suppose a wire is wound along a cone in such a way that its path is described by

$$r(t) = (t^2, t\cos t, t\sin t), \quad 1 \le t \le 4$$

If the charge density (per unit length) along the wire is given by

$$\rho(x,y,z) = \frac{y^2+z^2}{\sqrt{x}}$$

find the total charge in the wire.

$1 \le t \le 4$

$r(t) = (t^2, t\cos t, t\sin t)$

$r'(t) = (2t, \cos t - t\sin t, t\cos t + \sin t)$

$\|r'(t)\| = \sqrt{4t^2 + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t}$

$= \sqrt{4t^2 + \cos^2 t + \sin^2 t + t^2\sin^2 t + t^2\cos^2 t}$

$= \sqrt{4t^2 + 1 + t^2} = \sqrt{5t^2+1}$

$f(r(t)) = \dfrac{t^2\cos^2 t + t^2\sin^2 t}{\cancel{t}}$

$= t\cos^2 t + t\sin^2 t = t$

$\int f(r(t)) \cdot \|r'(t)\| \, dt$

$$\int_1^4 t \cdot \sqrt{5t^2+1} \, dt$$

10/10

$u = 5t^2 + 1$

$du = 10t \, dt$

$\frac{du}{10} = t\,dt \qquad \frac{1}{10}\int u^{\frac{1}{2}} du = \frac{1}{10} \cdot \frac{2}{3} u^{3/2}$

$= \left[\frac{1}{15}(5t^2+1)^{3/2} \Big|_1^4 \right]$

$\boxed{= \frac{1}{15}(81)^{3/2} - \frac{1}{15}(6)^{3/2}}$

Problem 3 [10pt]. Suppose a magnetic field will impart on some object a force

$$F(x,y,z) = \Big(-ye^z, xe^z, e^{x^2} - \cos(z^2)\Big)$$

when the object is at the point (x, y, z) in space. Compute the work done by this force in moving the object one full rotation counterclockwise (as viewed from above) around the circle

$$x^2 + y^2 = 1$$

$r(t) = (\cos t, \sin t, 0) \quad 0 \le t \le 2\pi$

$r'(t) = (-\sin t, \cos t, 0)$

in the xy-plane.

$$\int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t, e^{\cos^2 t} - 1\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$$

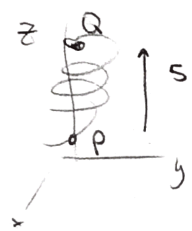
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t + 0\, dt = \int_0^{2\pi} 1\, dt = t\Big|_0^{2\pi}$$

$$\boxed{= 2\pi}$$ ✓

Problem 4 [10pt]. Let C be a helix of radius 4, centered along the z-axis, that makes 100 complete rotations clockwise while going up 5 units in the z direction, starting from the point $(0,3,1)$. Compute the following integral:

$$\int_C \overset{F_1}{\sin(y)}\,dx + \overset{F_2}{\left(x\cos(y) - \frac{1}{z}\right)}dy + \overset{F_3}{\frac{y}{z^2}}\,dz.$$

(Hint: You don't need to parametrize C. Just check if the filed is conservative and employ the properties of conservative vector fields).



$P(0,3,1)$

$Q(0,3,6)$

$F = \langle \sin y,\ x\cos(y) - \frac{1}{z},\ \frac{y}{z^2} \rangle$

$\text{curl}\,F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$\int F_1 = x\sin y$ $\quad \int F_2 = x\sin(y) - \frac{y}{z}$ $\quad \int F_3 = \frac{-y}{z}$

potential $f = x\sin y - \frac{y}{z}$ ✓

? $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$ $\quad \frac{1}{z^2} = \frac{1}{z^2}$ ✓

? $\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$ $\quad 0 = 0$ ✓

? $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ $\quad \cos y = \cos y$ ✓

F is conservative, path independent

$\int_C F = f(Q) - f(P)$ ✓

$= f(0,3,6) - f(0,3,1)$

$= (0 - \frac{3}{6}) - (0 - \frac{3}{1})$ ✓

$-\frac{1}{2} + 3 = \boxed{2\frac{1}{2}}$ ✓