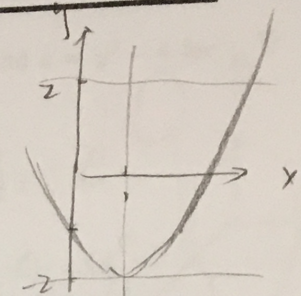


**Problem 1** **[10pt].** Compute the double integral

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$$\int_{-2}^{2}\int_{1}^{1+\sqrt{2+y}} x^2 \, dx dy.$$

Hint: First change the order of integration. $dx\,dy \Rightarrow dy\,dx$

$1 \le x \le 1+\sqrt{2+y}$ $\quad -2 \le y \le 2$

$x = 1+\sqrt{2+y}$ $\quad 2+y = (x-1)^2 = x^2-2x+1$

$y = x^2-2x-1$ $\quad x^2-2x-1 \le y \le 2$

$x = 1+\sqrt{2+2} = 3$ $\quad 1 \le x \le 3$

$$\int_1^3\int_{x^2-2x-1}^{2} x^2 \, dy\, dx = \int_1^3 x^2 y \Big|_{y=x^2-2x-1}^{2} dx$$

$$= \int_1^3 x^2(2-x^2+2x+1)\,dx = \int_1^3 -x^4+2x^3+3x^2\,dx$$

$$= -\frac{1}{5}x^5+\frac{1}{2}x^4+x^3\Big|_{x=1}^{3} = -\frac{243}{5}+\frac{81}{2}+27+\frac{1}{5}-\frac{1}{2}-1$$

$$= -\frac{242}{5}+40+26 = \boxed{\frac{88}{5}}$$

**Problem 2 [10pt].** Find the volume of the region enclosed by $z = 4 - y^2$ and $z = y^2 - 4$ for $0 \le x \le 3$.

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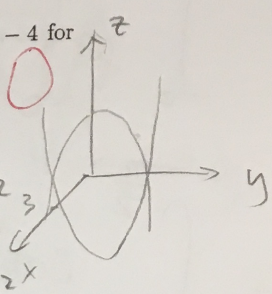



$4-y^2 = y^2-4 \quad y = 4 \quad y_1 = -2 \quad y_2 = 2$

$0 \le x \le 3 \quad -2 \le y \le 2 \quad y^2-4 \le z \le 4-y^2$

$$\text{volume} = \iiint_W 1\,dV = \int_0^3 \int_{-2}^{2} \int_{y^2-4}^{4-y^2} 1\,dz\,dy\,dx$$

$$= \int_0^3 \int_{-2}^{2} z \Big|_{z=y^2-4}^{4-y^2} dy\,dx = \int_0^3 \int_{-2}^{2} 4-y^2-y^2+4\,dy\,dx$$

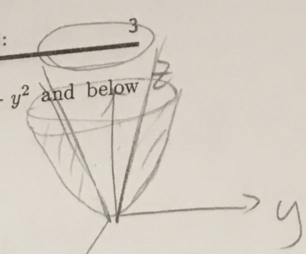
$$= \int_0^3 \int_{-2}^{2} 8-2y^2\,dy\,dx = \int_0^3 8y - \frac{2}{3}y^3 \Big|_{y=-2}^{2} dx$$

$$= \int_0^3 8*4 - \frac{2}{3}(8+8)\,dx = \int_0^3 32 - \frac{32}{3}\,dx$$

$$= \int_0^3 \frac{64}{3}\,dx = \frac{64}{3}x \Big|_{x=0}^{3} = \boxed{64}$$

**Problem 3 [10pt].** Let $W$ be the region in the first octant above $z = x^2 + y^2$ and below $z = 2\sqrt{x^2+y^2}$. Compute

$$\iiint_W x\, dV.$$

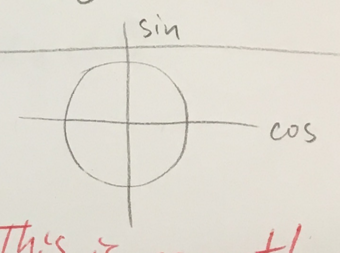



$x^2+y^2 \le z \le 2\sqrt{x^2+y^2}$ easier to use cylindrical coordinates $dV = r\,dr\,dz\,d\theta$

$0 \le \theta \le \frac{\pi}{2}$ $\qquad 0 \le z \le 4$

$r^2 = 2r \qquad r_1 = 2 \quad r_2 = 0 \quad 0 \le r \le 2$

$$\int_0^{\frac{\pi}{2}} \int_0^4 \int_{\sqrt{z}}^{\frac{z}{2}} r\cos\theta \, r \, dr\,dz\,d\theta$$




$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^2\cos\theta\, z \Big|_{z=r^2}^{2r} dr\,d\theta$$

This is correct!

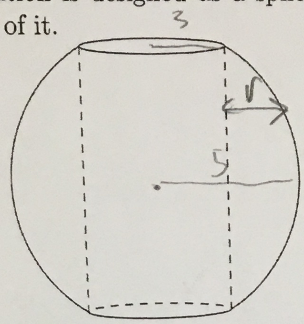
$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^2\cos\theta(2r - r^2)\, dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^2 2r^3\cos\theta - r^4\cos\theta \; dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}r^4\cos\theta - \frac{1}{5}r^5\cos\theta \Big|_{r=0}^{2} d\theta = \int_0^{\frac{\pi}{2}} 8\cos\theta - \frac{32}{5}\cos\theta \, d\theta$$

$$= 8\sin\theta - \frac{32}{5}\sin\theta \Big|_{\theta=0}^{\frac{\pi}{2}} = (1-0)\left(\frac{8}{5}\right) = \boxed{\frac{8}{5}}$$

(See the left side of the page)

**Problem 4 [10pt].** A space station is designed as a sphere of radius 5 km with a cylinder of radius 3 km removed from the center of it.




$x^2+y^2=9$

$r=5 \quad x^2+y^2+z^2=25$

$z^2=25-$

$=16$

The heat density in the air in the space station is given by $\delta(x,y,z) = \frac{1}{x^2+y^2+2z^2}$ where x, y, and z are measured in km, with $(0,0,0)$ located at the center of the sphere. Compute the total amount of heat in the air inside the space station. Hint: In cylindrical coordinates $(\theta, r, z)$, the region is $r$-simple; $\theta$ and $z$ are bounded by constants and the bound for $r$ depends on $z$. This should help to determine the order of integration.

$r\,dr\,dz\,d\theta$

$$\iiint_W \delta(x,y,z)\,dV \qquad -4 \le z \le 4$$

$$= \int_0^{2\pi} \int_{-4}^{4} \int_3^{\sqrt{25-z^2}} \frac{1}{r^2+2z^2}\, r\,dr\,dz\,d\theta$$

+4

$$= \int_0^{2\pi} \int_{-4}^{4} \int_3^{\sqrt{25-z^2}}$$