Math 32B-2 Yeliussizov. Midterm 1

Exam time: 5:00-6:30 PM, April 24, 2017

Last name: Wu First name: Vince

Student ID: 304805121

Discussion: BOOZER 2A Tue, 2B Thu, KALYANSWAMY 2C Tue, 2D Thu; GUO 2E Tue, 2F Thu

There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

P 1 (10)	P 2 (10)	P 3 (10)	P 4 (15)	P 5 (15)	Total (60 pt)
10	10	10	13	15	58

Problem 1. (10 points) Evaluate the double integral over the given rectangular domain in the xy-plane

$$\iint_R (y^3 + 2 + 3y\sqrt{1+xy})\, dA, \qquad R = [0,1] \times [-1,1].$$

$u = xy$

$du = y\, dx$

$dx = \frac{1}{y}\, du$

$\int 3y\sqrt{1+xy}\, dx$

$= 3\int (1+u)^{\frac{1}{2}}\, du$

$= \cancel{3}\left(\frac{2}{3}\right)(1+u)^{\frac{3}{2}}$

$= 2(1+xy)^{\frac{3}{2}}$

$$= \int_{-1}^{1}\int_{0}^{1} \left(y^3 + 2 + 3y\sqrt{1+xy}\right) dx\, dy$$

$$= \int_{-1}^{1} xy^3 + 2x + 2(1+xy)^{\frac{3}{2}} \Big|_{x=0}^{1}\, dy$$

$$= \int_{-1}^{1} y^3 + \cancel{2} + 2(1+y)^{\frac{3}{2}} - \cancel{2}\, dy$$

$$= \frac{1}{4}y^4 + 2\left(\frac{2}{5}\right)(1+y)^{\frac{5}{2}} \Big|_{-1}^{1}$$

$$= \cancel{\frac{1}{4}} + \frac{4}{5}\left(2^{\frac{5}{2}}\right) - \cancel{\frac{1}{4}}$$

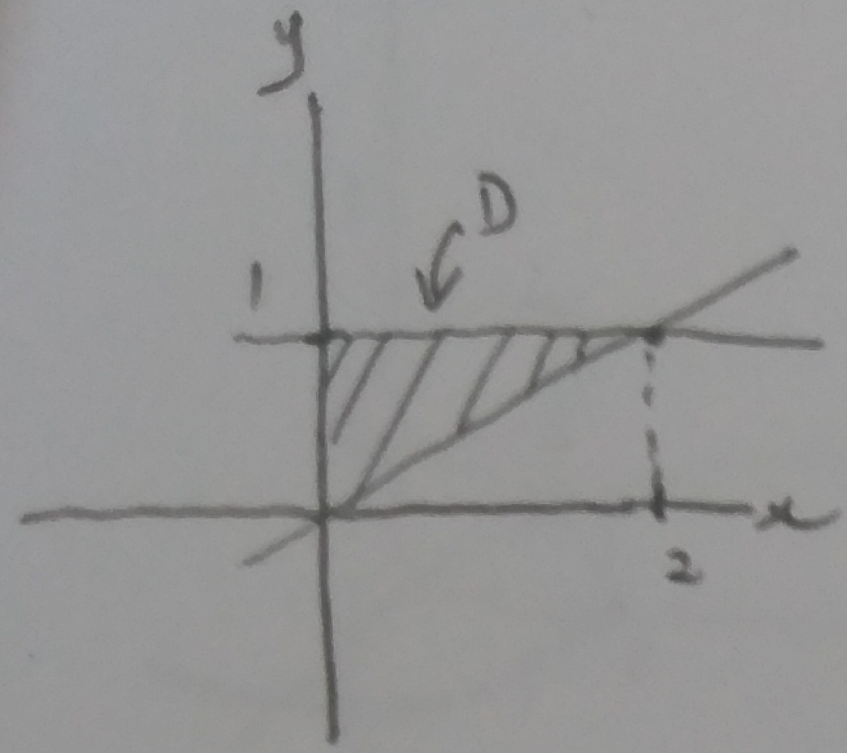
$$= \frac{4}{5}\left(2^{\frac{5}{2}}\right)$$

$$= \frac{4}{5}\left(2^{2} \cdot 2^{\frac{1}{2}}\right)$$

$$= \boxed{\frac{16}{5}\sqrt{2}}$$

10

Problem 2. (10 points) Evaluate the integral $\int_0^2 \int_{x/2}^1 e^{y^2}\, dy\, dx$ by changing the order of integration.



$y = \frac{1}{2}x$

$x = 2y$

$0 \le y \le 1$

$0 \le x \le 2y$

$$\int_0^2 \int_{x/2}^1 e^{y^2}\, dy\, dx$$

$$= \int_0^1 \int_0^{2y} e^{y^2}\, dx\, dy$$

$$= \int_0^1 2y e^{y^2}\, dy$$

$$= e^{y^2} \Big|_0^1$$

$$= \boxed{e - 1}$$

$u = y^2$

$du = 2y\, dy \Rightarrow$

$dy = \frac{1}{2y}\, du$

$\int 2y e^{y^2}\, dy$

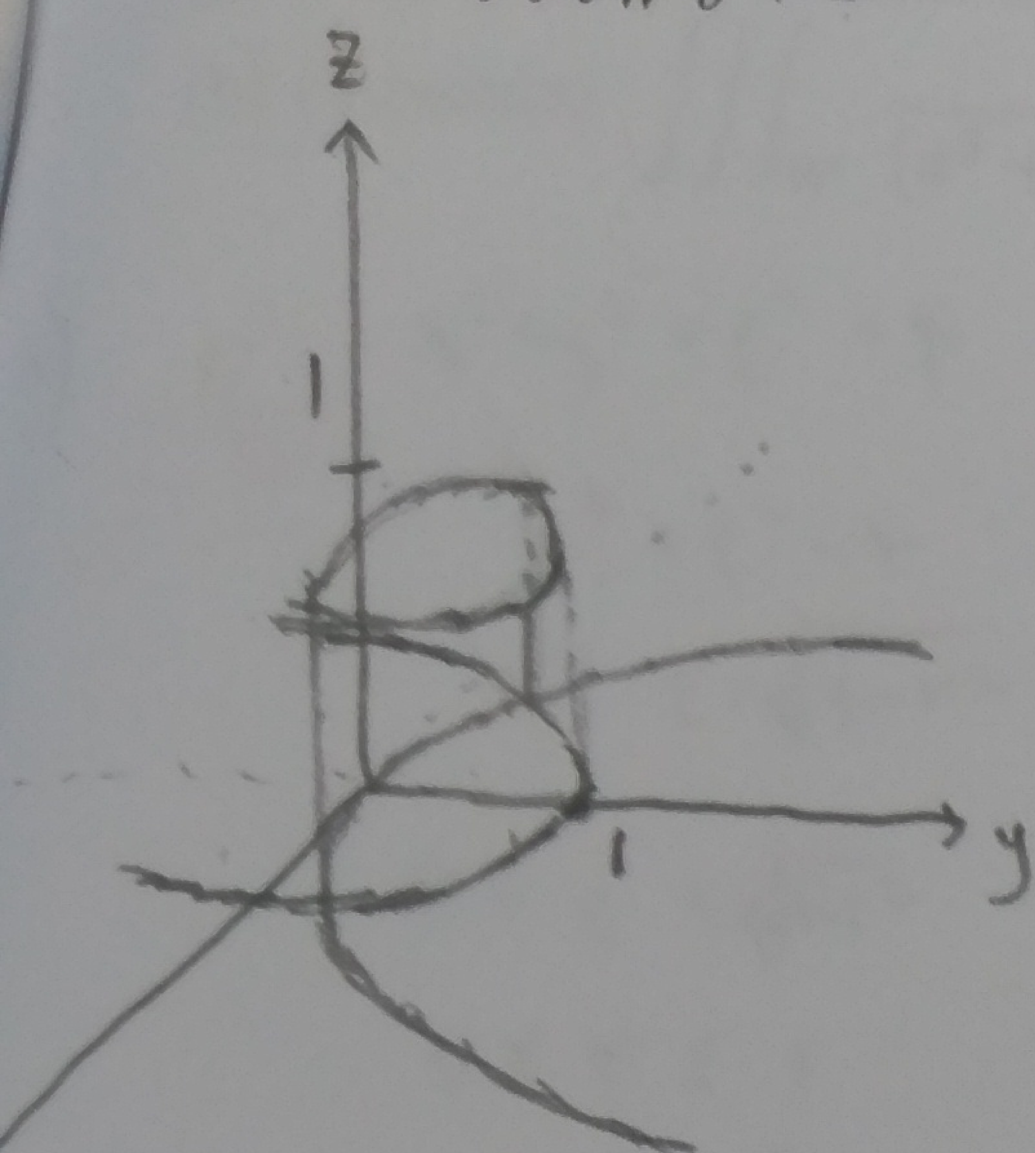
$= \int e^u\, du$

$= e^u$

$= e^{y^2}$

Problem 3. (10 points) Let W be the region enclosed by $y = x^2$, $y = 1 - x^2$, $z = 1$, and $z + y = 0$. Evaluate $\iiint_W \frac{2}{y+1}\, dV$.

$z = -y$



$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{x^2}^{1-x^2} \int_{-y}^{1} \frac{2}{y+1}\, dz\, dy\, dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{x^2}^{1-x^2} \frac{2}{y+1} + \frac{2y}{y+1}\, dy\, dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{x^2}^{1-x^2} \frac{2\cancel{(1+y)}}{\cancel{y+1}}\, dy\, dx$$

$x^2 = 1 - x^2$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm \frac{1}{\sqrt{2}}$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 2(1-x^2) - 2x^2\, dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 2 - 2x^2 - 2x^2\, dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 2 - 4x^2\, dx$$

$$= 2x - \frac{4}{3}x^3 \Big|_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}$$

$$= 2\left(\frac{1}{\sqrt{2}}\right) - \frac{4}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - 2\left(-\frac{1}{\sqrt{2}}\right) + \frac{4}{3}\left(-\frac{1}{\sqrt{2}}\right)^3$$

$$= 2\left(\frac{1}{\sqrt{2}}\right) - \frac{4}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + 2\left(\frac{1}{\sqrt{2}}\right) - \frac{4}{3}\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= \frac{4}{\sqrt{2}} - \frac{8}{3\sqrt{8}}$$

$$= \frac{4}{\sqrt{2}} - \frac{\cancel{8}^{4}}{36\sqrt{2}}$$

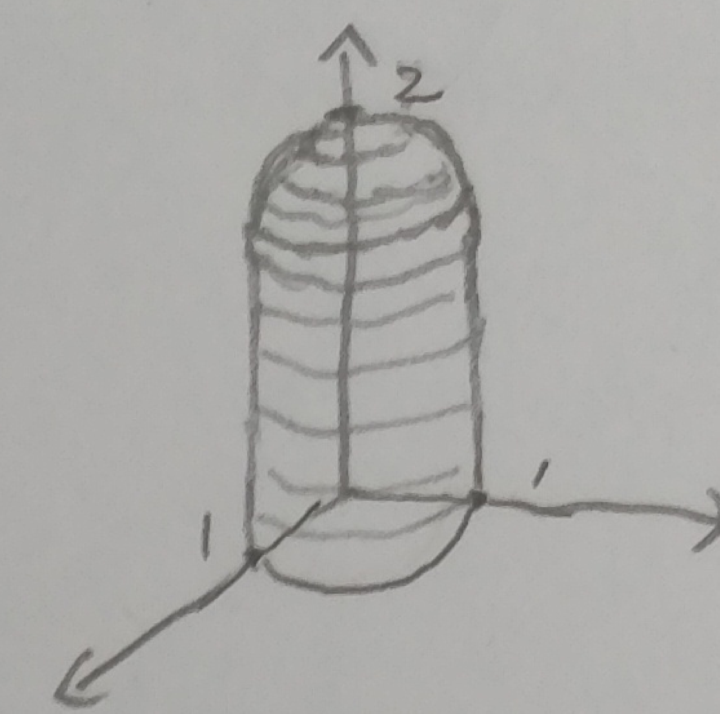
$$= \frac{8}{3\sqrt{2}}$$

16
$\times 24$
64

Problem 4. (15 points) Let W be the region in the $\boxed{\text{first octant}}$ $x, y, z \geq 0$ inside the cylinder $x^2+y^2=1$ and bounded by the sphere $x^2+y^2+z^2=4$. Evaluate

(a) (7 points) the volume of W via cylindrical coordinates

(b) (8 points) $\displaystyle\iiint_W \frac{z}{(x^2+y^2+z^2)^{3/2}}\, dV$ via spherical coordinates



a) 6

$x^2+y^2+z^2=4$

$r^2+z^2=4$

$z=\sqrt{4-r^2}$

$0 \leq z \leq \sqrt{4-r^2}$

$0 \leq r \leq 1$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r\sqrt{4-r^2}\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}(4-r^2)^{\frac{3}{2}} \Big|_{r=0}^{1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}(3)^{\frac{3}{2}} - \frac{1}{3}(4)^{\frac{3}{2}}\, d\theta$$

$$= \frac{\pi}{2}\left(\sqrt{3} - \frac{8}{3}\right)$$

$$= \boxed{\frac{\sqrt{3}\pi}{2} - \frac{8\pi}{3}}$$

$\int r\sqrt{4-r^2}\, dr$

$u = r^2$

$du = 2r\,dr \Rightarrow$

$dr = \frac{1}{2r}du$

$= \frac{1}{2}\int \sqrt{4-u}\, du$

$= \frac{1}{2}\frac{2}{3}(4-u)^{\frac{3}{2}}$

$= \frac{1}{3}(4-r^2)^{\frac{3}{2}}$

b) 7

$x^2+y^2+z^2=\rho^2$

$z=\rho\cos\phi$

$r=1$

$\rho\sin\phi=1$

$\rho=\frac{1}{\sin\phi}$

$(1)^2+(0)^2+z^2=4$

$1+z^2=4$

$z=\sqrt{3}$

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{6}}\int_0^{2} \frac{\rho\cos\phi}{(\rho^2)^{3/2}}\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{6}}\int_0^{2} \frac{\rho\cos\phi}{\rho^3}\, \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{6}} 2\cos\phi\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\cos^2\phi \Big|_{\phi=0}^{\frac{\pi}{6}} d\theta$$

$$= \frac{\pi}{4}\left(\frac{3}{4}-1\right)$$

$$= \frac{3\pi}{16} - \frac{\pi}{4}$$

$\int u\,dv = uv - \int v\,du$

$u=\cos\phi \quad dv=\sin\phi$

$du=-\sin\phi \quad v=-\cos\phi$

$\int\cos\phi\sin\phi = \cos^2\phi - \int\cos\phi\sin\phi$

$2\int\cos\phi\sin\phi = \cos^2\phi$

$= \frac{1}{2}\cos^2\phi$

$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\frac{1}{\sin\phi}} \cos\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\phi\sin\phi\left(\frac{1}{\sin\phi}\right) d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\phi \Big|_{\phi=\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta$$

$$= \frac{\pi}{2}\left(1-\frac{1}{2}\right)$$

$$= \frac{\pi}{4}$$

$\frac{3\pi}{16} - \frac{\pi}{4} + \frac{\pi}{4}$

$= \boxed{\cancel{\frac{3\pi}{16}}}$

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$\times 24$
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Problem 4. (15 points) Let W be the region in the **first octant** $x, y, z \geq 0$ inside the cylinder $x^2+y^2=1$ and bounded by the sphere $x^2+y^2+z^2=4$. Evaluate

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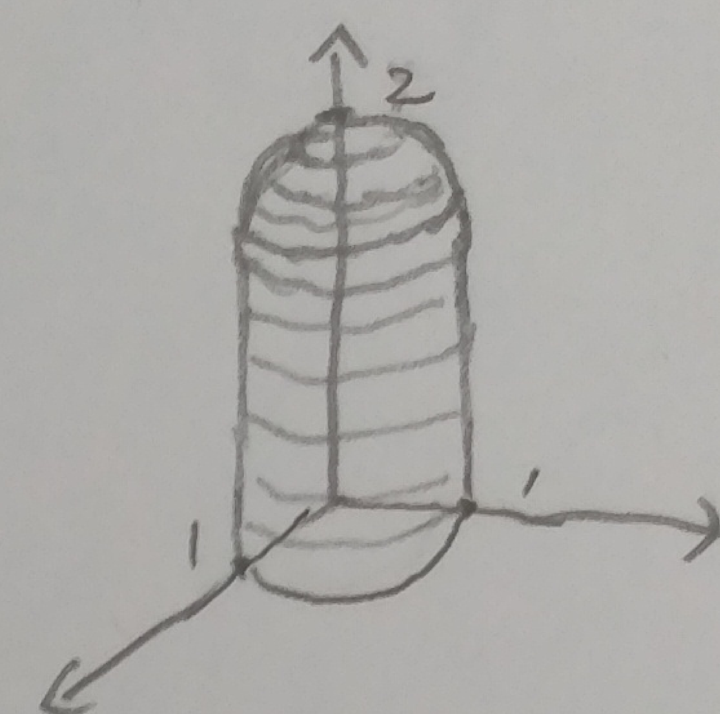

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$$=\frac{\pi}{2}\left(\sqrt{3}-\frac{8}{3}\right)$$

$$=\boxed{\frac{\sqrt{3}\pi}{2}-\frac{8\pi}{3}}$$

$u=r^2$

$du=2r\,dr \Rightarrow$

$dr=\frac{1}{2r}du$

$\int r\sqrt{4-r^2}\,dr$

$=\frac{1}{2}\int\sqrt{4-u}\,du$

$=\frac{1}{2}\cdot\frac{2}{3}(4-u)^{\frac{3}{2}}$

$=\frac{1}{3}(4-r^2)^{\frac{3}{2}}$

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$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{6}}\int_0^{2}\frac{\rho\cos\phi}{(\rho^2)^{3/2}}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{6}}\int_0^{2}\frac{\rho\cos\phi}{\rho^3}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

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$$=\frac{\pi}{4}\left(\frac{3}{4}-1\right)$$

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$\int u\,dv=uv-\int v\,du$

$u=\cos\phi \quad dv=\sin\phi$

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$\int\cos\phi\sin\phi=\cos^2\phi-\int\cos\phi\sin\phi$

$2\int\cos\phi\sin\phi=\cos^2\phi$

$=\frac{1}{2}\cos^2\phi$

$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\frac{1}{\sin\phi}}\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos\phi\sin\phi\left(\frac{1}{\sin\phi}\right)d\phi\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sin\phi\Big|_{\phi=\frac{\pi}{6}}^{\frac{\pi}{2}}\,d\theta$$

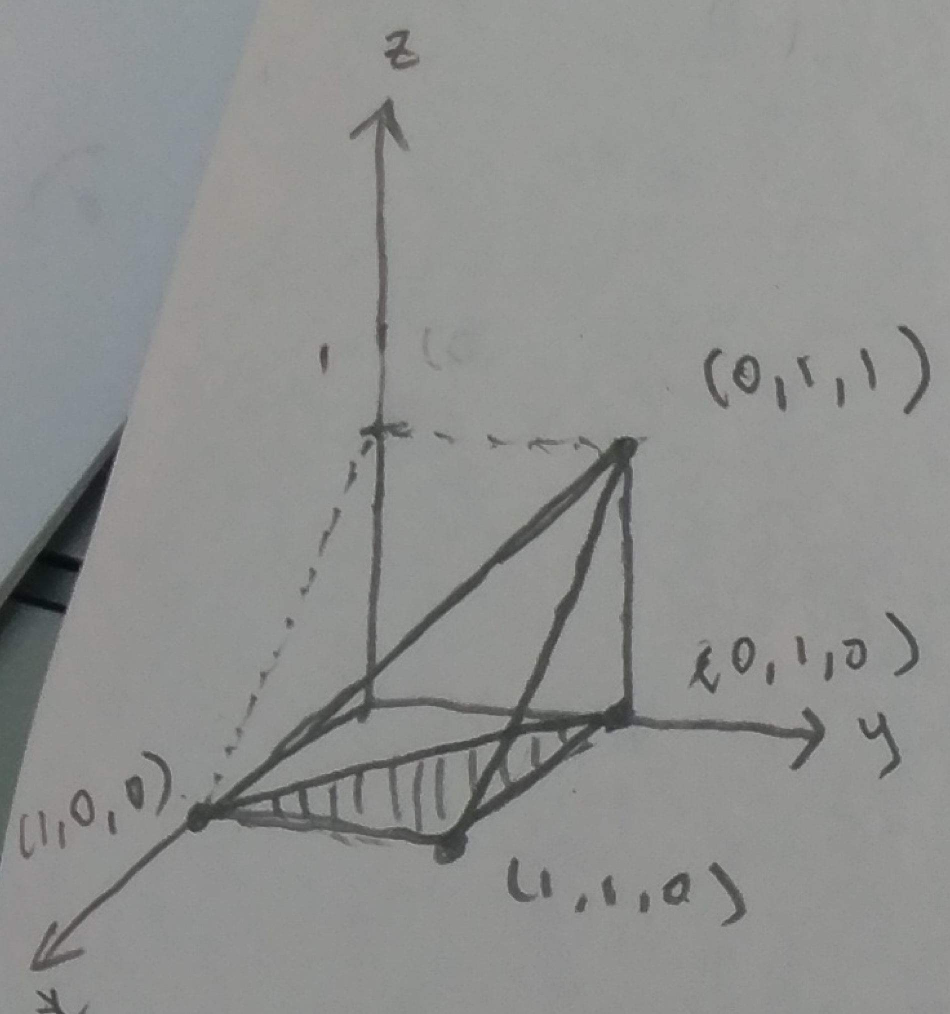
$$=\frac{\pi}{2}\left(1-\frac{1}{2}\right)$$

$$=\frac{\pi}{4}$$

$\frac{3\pi}{16}-\frac{\pi}{4}+\frac{\pi}{4}$

$=\boxed{\frac{3\pi}{16}}$

Problem 5. (15 points) Let W be the tetrahedron with vertices $(1,0,0)$, $(1,1,0)$, $(0,1,0)$, and $(0,1,1)$.
(a) (8 points) Find the volume of W.
(b) (7 points) Find the z-coordinate of the centroid of W (i.e., the center of mass assuming the mass density $\delta(x,y,z)=1$).



$0 \le x \le 1$

$-x+1 \le y \le 1$

$0 \le z \le -x+1$

a) $\int_0^1 \int_{-x+1}^1 \int_0^{-x+1} 1 \, dz \, dy \, dx$

$= \int_0^1 \int_{-x+1}^1 (-x+1) \, dy \, dx$

$= \int_0^1 (-x+1) - (-x+1)^2 \, dx$

$= \int_0^1 -x+1-(x^2-2x+1) \, dx$

$= \int_0^1 -x\cancel{+1} - x^2 + 2x \cancel{-1} \, dx$

$= \int_0^1 -x^2 + x \, dx$

$= -\frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_0^1 = -\frac{1}{3} + \frac{1}{2} = \boxed{\frac{1}{6}}$

b) $M = \iiint_W \delta(x,y,z) \, dV$

$= \iiint_W 1 \, dV = V = \frac{1}{6}$

$M_z = \iiint_W z\,\delta(x,y,z) \, dV$

$= \int_0^1 \int_{-x+1}^1 \int_0^{-x+1} z \, dz \, dy \, dx$

$= \int_0^1 \int_{-x+1}^1 \frac{1}{2}z^2 \Big|_{z=0}^{-x+1} dy \, dx$

$= \int_0^1 \int_{-x+1}^1 \frac{1}{2}(-x+1)^2 \, dy \, dx$

$= \int_0^1 \frac{1}{2}(-x+1)^2 - \frac{1}{2}(-x+1)^3 \, dx$

$= \int_0^1 \frac{1}{2}(x^2-2x+1) - \frac{1}{2}(-x+1)(x^2-2x+1) \, dx$

$= \int_0^1 \frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{2}(-x^3+2x^2-x+x^2-2x+1) \, dx$

$= \int_0^1 \cancel{\frac{1}{2}x^2} \cancel{-x} \cancel{+\frac{1}{2}} + \frac{1}{2}x^3 - x^2 + \frac{1}{2}x - \cancel{\frac{1}{2}x^2} \cancel{+x} \cancel{-\frac{1}{2}} \, dx$

$= \int_0^1 \frac{1}{2}x^3 - x^2 + \frac{1}{2}x \, dx$

$= \frac{1}{8}x^4 - \frac{1}{3}x^3 + \frac{1}{4}x^2 \Big|_0^1 = \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$

$= \frac{3}{24} - \frac{8}{24} + \frac{6}{24} = \frac{1}{24}$

$\bar{z} = \frac{M_z}{M} = \frac{\frac{1}{24}}{\frac{1}{6}} = \frac{\cancel{6}\,1}{\cancel{24}\,4} = \frac{1}{4}$