1. (10 pts)

(a) Find the area of the triangle with vertices $\overset{P=}{(-3,1,-4)}$, $\overset{Q=}{(2,0,-5)}$, and $\underset{R=}{(1,-3,-2)}$.

$\vec{PQ} = \langle 5, -1, -1 \rangle \qquad \vec{PR} = \langle 4, -4, 2 \rangle$

$$\vec{PQ} \times \vec{PR} = \det \begin{bmatrix} i & j & k \\ 5 & -1 & -1 \\ 4 & -4 & 2 \end{bmatrix} = \det \begin{bmatrix} -1 & -1 \\ -4 & 2 \end{bmatrix} i - \det \begin{bmatrix} 5 & -1 \\ 4 & 2 \end{bmatrix} j + \det \begin{bmatrix} 5 & -1 \\ 4 & -4 \end{bmatrix} k$$

$$= -6i - 14j - 16k = \langle -6, -14, -16 \rangle$$

$$[PQR] = \frac{1}{2} \| \vec{PQ} \times \vec{PR} \| = \frac{1}{2} \sqrt{(-6)^2 + (-14)^2 + (-16)^2}$$

$$= \frac{1}{2} \sqrt{36 + 196 + 256} = \frac{1}{2} \sqrt{488} = \boxed{\sqrt{122}}$$

(b) Find an equation for the plane that contains the triangle from part (a).

$\vec{n} = \vec{PQ} \times \vec{PR} = \langle -6, -14, -16 \rangle$

point $= (-3, 1, -4)$

$$\langle x+3, y-1, z+4 \rangle \cdot \langle -6, -14, -16 \rangle = 0$$

$$-6x - 14y - 16z = 18 - 14 + 64$$

$$6x + 14y + 16z = -68$$

$$\boxed{3x + 7y + 8z = -34}$$

2. (10 pts) Two forces are acting on an object:

- a force of $\langle 3, 5, -2\rangle$ is applied at the point $(-1, 2, 0)$;
- a force of $\langle -4, 5, 3\rangle$ is applied at the point $(-2, 3, 1)$.

(a) Find the total torque, relative to the origin, caused by these two forces.

$$\tau_1 = r_1 \times F_1 = \langle -1, 2, 0\rangle \times \langle 3, 5, -2\rangle$$

$$= \det\begin{bmatrix} i & j & k \\ -1 & 2 & 0 \\ 3 & 5 & -2 \end{bmatrix} = \det\begin{bmatrix} 2 & 0 \\ 5 & -2 \end{bmatrix} i - \det\begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix} j + \det\begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix} k$$

$$= -4i - 2j - 11k = \langle -4, -2, -11\rangle \checkmark$$

$$\tau_2 = r_2 \times F_2 = \langle -2, 3, 1\rangle \times \langle -4, 5, 3\rangle$$

$$= \det\begin{bmatrix} i & j & k \\ -2 & 3 & 1 \\ -4 & 5 & 3 \end{bmatrix} = \det\begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix} i - \det\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} j + \det\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix} k$$

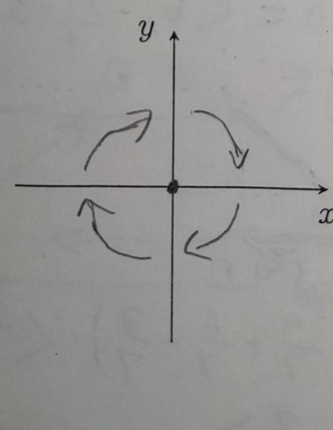
$$= 4i + 2j + 2k = \langle 4, 2, 2\rangle \checkmark$$

$$\tau_1 + \tau_2 = \langle -4, -2, -11\rangle + \langle 4, 2, 2\rangle = \boxed{\langle 0, 0, -9\rangle} \checkmark$$

(b) In what direction would the object rotate? You can indicate your answer by drawing arrows on the axes provided below.

Clockwise from this perspective. ✓

• = axis of rotation



✓

3. (10 pts) Consider the line defined by

$$\vec{r}(t) = \langle -3+t, -3+6t, 1-2t \rangle$$

and the plane

$$6x - 2y - 3z = 6.$$

(a) (2 pts) Explain briefly why the line is parallel to the plane.

direction vector for $\vec{r}(t) = \langle 1, 6, -2 \rangle$

normal vector for plane $= \langle 6, -2, -3 \rangle$

$\langle 1, 6, -2 \rangle \cdot \langle 6, -2, -3 \rangle = 6 - 12 + 6 = 0$ so they are orthogonal to each other.

Since the direction vector (for the line) and the normal vector for the plane are orthogonal to each other, the line $\vec{r}(t)$ is parallel to the plane $6x - 2y - 3z = 6$.

(b) (2 pts) Write down any point P on the plane, any point Q on the line, and find the vector $\overrightarrow{PQ}$:

P: $(1, 0, 0)$

Q: $(-3, -3, 1)$

$\overrightarrow{PQ}$: $\langle 4, -3, 1 \rangle$

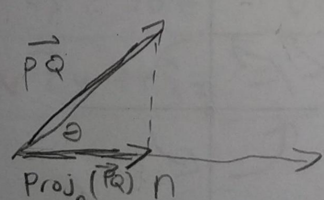
(c) (4 pts) Compute the projection of $\overrightarrow{PQ}$ onto the normal vector of the plane.

$\vec{n} = \langle 6, -2, -3 \rangle \quad \vec{PQ} = \langle 4, -3, 1 \rangle$

$e_n = \langle \frac{6}{7}, -\frac{2}{7}, -\frac{3}{7} \rangle$

$\mathrm{Proj}_{\vec{n}}(\vec{PQ}) = \|\vec{PQ}\| \cos\theta \cdot e_n$

$= \frac{\vec{PQ} \cdot \vec{n}}{\|n\|} e_n = (\vec{PQ} \cdot e_n) \cdot e_n$



$\mathrm{Proj}_{\vec{n}}(\vec{PQ}) = (\vec{PQ} \cdot e_n) \cdot e_n = \left(\frac{24}{7} + \frac{6}{7} - \frac{3}{7}\right) \cdot \langle \frac{6}{7}, -\frac{2}{7}, -\frac{3}{7} \rangle$

$= \frac{27}{7} \langle \frac{6}{7}, -\frac{2}{7}, -\frac{3}{7} \rangle = \boxed{\langle \frac{16.2}{7}, -\frac{54}{7}, -\frac{81}{7} \rangle}$

(continued on next page . . .)

(d) (2 pts) Compute the magnitude of the projection you computed in part (c).

2

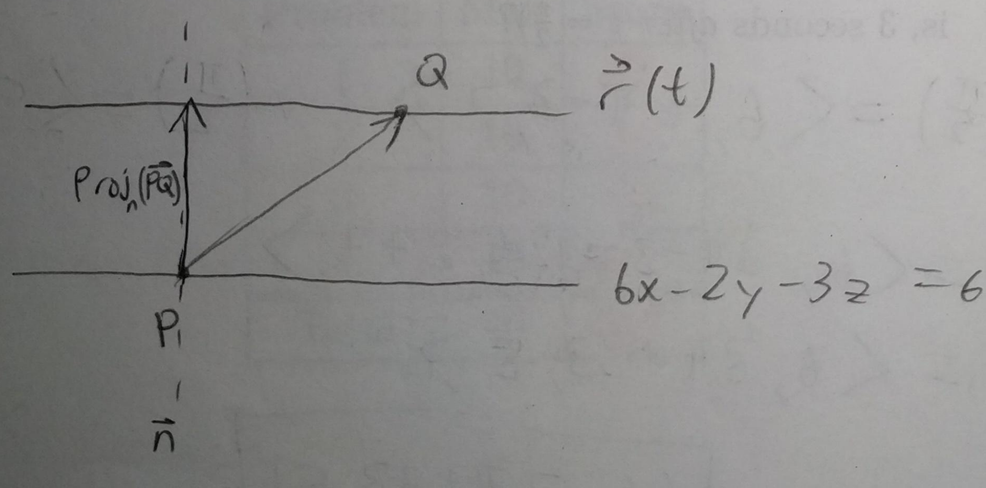
$$\sqrt{\left(\frac{162}{7}\right)^2 + \left(-\frac{54}{7}\right)^2 + \left(-\frac{81}{7}\right)^2}$$

= some horrible fraction I can't calculate by hand.

Bonus: (2 pts) What does the magnitude that you computed in part (d) represent?

2

(shortest) distance between the line $\vec{r}(t)$ and the plane $6x-2y-3z=6$



4. (10 pts) For a physics experiment, you build a machine that subjects an object to an acceleration of

$$\vec{a}(t) = \langle 8\cos(2t), 12\sin(2t), \cos(t) \rangle.$$

The object starts from rest (i.e., with zero velocity) at the position $(2, -3, 1)$.

(a) Find the position of the object as a function of t.

$v(t) = \int a(t)\,dt = \langle 4\sin(2t), -6\cos(2t), \sin(t) \rangle + C$

$v(0) = \langle 0, -6, 0 \rangle + C = \langle 0, 0, 0 \rangle \Rightarrow C = \langle 0, 6, 0 \rangle$

$v(t) = \langle 4\sin(2t), -6\cos(2t) + 6, \sin(t) \rangle$

$r(t) = \int v(t)\,dt = \langle -2\cos(2t), -3\sin(2t) + 6t, -\cos(t) \rangle + C$

$r(0) = \langle -2, 0, -1 \rangle + C = \langle 2, -3, 1 \rangle \Rightarrow C = \langle 4, -3, 2 \rangle$

$r(t) = \boxed{\langle -2\cos(2t) + 4, -3\sin(2t) + 6t - 3, -\cos(t) + 2 \rangle}$

(b) Suppose you turn off the machine at time $t = \frac{\pi}{2}$, so that from then on the object experiences no acceleration, and thus moves in a straight line in the same direction it was going at that instant. (Assume this experiment is taking place in space, so there's no gravity or friction.) What will be the final position of the object 3 seconds later (that is, 3 seconds *after* $t = \frac{\pi}{2}$)?

$r(\frac{\pi}{2}) = \langle 6, 3\pi - 3, 2 \rangle \quad v(\frac{\pi}{2}) = \langle 0, 12, 1 \rangle$

$r(t) = \langle 6, 3\pi - 3 + 12t, 2 + t \rangle$

$r(3) = \langle 6, 3\pi + 33, 5 \rangle$

$\boxed{(6, 3\pi + 33, 5)}$