MATH 32A MIDTERM 2

1 hour

Section	UID	First Name, Last Name
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Pro. 1	Pro. 2	Pro. 3	Pro.4	Total
20	20	20	36	96

Curvature formula:

$$\kappa(t) = \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3}$$

Problem 1. (20 pts) Find the path length covered by the monement $\vec{r}(t) = < 2t+1, \ln t, t^2+2 >$ for $t \in [1,2]$

$\vec{v}(t) = \vec{r}'(t) = < 2 , \frac{1}{t} , 2t >$

$\|\vec{r}'(t)\| = \sqrt{4+\frac{1}{t^2}+4t^2} = \sqrt{(2t+\frac{1}{t})^2} = 2t+\frac{1}{t}$

$S = \int_1^2 \|\vec{r}'(t)\| dt = \int_1^2 (2t+\frac{1}{t}) dt$

$= \left(\frac{2t^2}{2} + \ln t\right)\Big|_1^2$

$= (t^2 + \ln t)\Big|_1^2$

$= (4+\ln 2) - (1+\ln 1)$

$= 3 + \ln 2$

20

Problem 2. (20 pts) Find an arc length parameterization $\vec{r_1}(s)$ for the movement $\vec{r}(t) = < \frac{4}{3}t^{\frac{3}{2}} + 1, t^2 + 5, t >$, where $t \in [0, +\infty)$.

Hints: The inverse function of $f(x) = x^2 + x$ over $[0, +\infty)$ is $g(y) = \sqrt{y + \frac{1}{4}} - \frac{1}{2}$. And you do not need to develope the expression $(...)^{\frac{3}{2}}$.

$\frac{3}{2} \times \frac{4}{3}\sqrt{t}$ $\quad 2\sqrt{t}$

$$\vec{r}'(t) = \langle 2t^{\frac{1}{2}}, 2t, 1 \rangle$$

$$= \langle 2\sqrt{t}, 2t, 1 \rangle$$

$2x + 1$

$$\|\vec{r}'(t)\| = \sqrt{4t + 4t^2 + 1}$$

$$= \sqrt{(2t+1)^2}$$

$$= 2t + 1$$

$$S(t) = \int \|\vec{r}'(t)\| \, dt$$

$$= \int (2t + 1)\, dt$$

$$= t^2 + t.$$

$$s^{-1}(w) = \sqrt{w + \frac{1}{4}} - \frac{1}{2}$$

$$\vec{r_1}(w) = \vec{r}(s^{-1}(w)) = \langle \frac{4}{3}\left(\sqrt{w+\frac{1}{4}} - \frac{1}{2}\right)^{\frac{3}{2}} + 1, \left(\sqrt{w+\frac{1}{4}} - \frac{1}{2}\right)^2 + 5, \sqrt{w+\frac{1}{4}} - \frac{1}{2} \rangle$$

$$= \langle \frac{4}{3}\left(\sqrt{w+\frac{1}{4}} - \frac{1}{2}\right)^{\frac{3}{2}} + 1, w + \frac{11}{2} - \sqrt{w+\frac{1}{4}}, \sqrt{w+\frac{1}{4}} - \frac{1}{2} \rangle$$

Problem 3. (20 pts) Let $\vec{r}(t)$ be a movement in $\mathbb{R}^3$. We know that $\vec{r}(0) =< 1, 2, 3 >$, $\vec{v}(0) =< 2, 0, 1 >$ and $\vec{a}(t) =< \sin t, 1, t >$. Use net change formula to find $\vec{v}(t)$ and $\vec{r}(t)$.

$$\vec{v}(t) = \int \vec{a}(t)\, dt$$

$$= \langle -\cos t, t, \frac{t^2}{2} \rangle + \langle C_1, C_2, C_3 \rangle$$

$$\vec{v}(0) = \langle -1, 0, 0 \rangle + \langle C_1, C_2, C_3 \rangle = \langle 2, 0, 1 \rangle$$

$$\Rightarrow \quad C_1 = 3 \quad C_2 = 0 \quad C_3 = 1$$

$$\vec{v}(t) = \langle -\cos t, t, \frac{t^2}{2} \rangle + \langle 3, 0, 1 \rangle$$

$$= \langle -\cos t + 3, t, \frac{t^2}{2} + 1 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\, dt$$

$$= \langle -\sin t + 3t, \frac{t^2}{2}, \frac{t^3}{6} + t \rangle + \langle C_4, C_5, C_6 \rangle$$

$$\vec{r}(0) = \langle 0, 0, 0 \rangle + \langle C_4, C_5, C_6 \rangle = \langle 1, 2, 3 \rangle$$

$$\Rightarrow \quad C_4 = 1 \quad C_5 = 2 \quad C_6 = 3$$

$$\vec{r}(t) = \langle -\sin t + 3t, \frac{t^2}{2}, \frac{t^3}{6} + t \rangle + \langle 1, 2, 3 \rangle$$

$$= \langle -\sin t + 3t + 1, \frac{t^2}{2} + 2, \frac{t^3}{6} + t + 3 \rangle$$

Problem 4. (40 pts) We consider $\vec{r}(t) = < \sin 2t, \cos 2t, t >$.

(1) Compute the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$
(2) Compute the unit tangent vector $\vec{T}(t)$
(3) Compute the curvature $\kappa(t)$
(4) Decompose $\vec{a}(t) = a_T\vec{T} + a_N\vec{N}$ into sum of tangent component and normal component (You need to write down the expressions of a_T, a_N and $\vec{N}$ for your final answer).
(5) Find the binormal vector $\vec{B}(t)$.
(6) Write down an equation of the oscillating plane at the moment t.
(7) Find the curvature center $Q(t)$.

(1) $\vec{v}(t) = \vec{r}\,'(t) = \langle 2\cos 2t, -2\sin 2t, 1 \rangle$

$\vec{a}(t) = \vec{r}\,''(t) = \langle -4\sin 2t, -4\cos 2t, 0 \rangle$

$0 - - 4\sin 2t$

(2) $\|\vec{v}(t)\| = \sqrt{4\cos^2 2t + 4\sin^2 2t + 1} = \sqrt{5}$

$-(8\cos^2 2t + 8\sin^2 2t)$

$\vec{T} = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|} = \dfrac{\langle 2\cos 2t, -2\sin 2t, 1 \rangle}{\sqrt{5}}$

-8

$= \langle \frac{2}{\sqrt{5}}\cos 2t, -\frac{2}{\sqrt{5}}\sin 2t, \frac{1}{\sqrt{5}} \rangle$

(3) $\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos 2t & -2\sin 2t & 1 \\ -4\sin 2t & -4\cos 2t & 0 \end{vmatrix} = 4\cos 2t\,\hat{i} - 4\sin 2t\,\hat{j} - 8\hat{k}$

$= \langle 4\cos 2t, -4\sin 2t, -8 \rangle$

$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{16\cos^2 2t + 16\sin^2 2t + 64} = \sqrt{80} = 4\sqrt{5}$

$\|\vec{r}\,'(t)\| = \sqrt{5}$

$\kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \dfrac{4\sqrt{5}}{5\sqrt{5}} = \dfrac{4}{5}$

(4) $a_{\vec{T}} = \vec{a} \cdot \vec{T} = 0$

$a_{\vec{N}}\vec{N} = \vec{a}(t) - a_{\vec{T}}\vec{T} = \langle -4\sin 2t, -4\cos 2t, 0 \rangle - \langle 0, 0, 0 \rangle$

$= \langle -4\sin 2t, -4\cos 2t, 0 \rangle$

$a_{\vec{N}} = \|a_{\vec{N}}\vec{N}\| = \sqrt{16\sin^2 2t + 16\cos^2 2t + 0} = \sqrt{16} = 4$

$\vec{N} = \dfrac{a_{\vec{N}}\vec{N}}{\|a_{\vec{N}}\vec{N}\|} = \dfrac{\langle -4\sin 2t, -4\cos 2t, 0 \rangle}{4}$

$= \langle -\sin 2t, -\cos 2t, 0 \rangle$

$\vec{a} = a_{\vec{T}}\vec{T} + a_{\vec{N}}\vec{N} = 0 \cdot \langle \frac{2}{\sqrt{5}}\cos 2t, -\frac{2}{\sqrt{5}}\sin 2t, \frac{1}{\sqrt{5}} \rangle + 4\langle -\sin 2t, -\cos 2t, 0 \rangle$