Math 32A, Lecture 1
Multivariable Calculus

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name:

UID:

Section:

Question	Points	Score
1	10	7
2	10	3
3	10	4
4	10	7
5	10	6
Total:	50	27

$\kappa(s) = \frac{\|r'(s) \times r''(s)\|}{\|r'(s)\|^3}$

Problem 1.
Consider the space curve $\mathbf{r}(s) = \langle \frac{1}{3}(1+s)^{\frac{3}{2}}, \frac{1}{3}(1-s)^{\frac{3}{2}}, \frac{s}{\sqrt{2}} \rangle$. →

(a) [3pts.] Show that $\mathbf{r}(s)$ is a unit speed curve.

(b) [4pts.] Find the curvature of $\mathbf{r}(s)$ at $s = 0$, and the unit normal vector $\mathbf{N}$ to the curve at this point. [Hint: In light of part (a), there is a fast way to do this.]

(c) [3pts.] Find the osculating plane to $\mathbf{r}(s)$ at $s = 0$, and say what it represents geometrically. Best contacts the curve

a) For a unit speed curve, $\|r'(s)\| = 1$ always

$r'(s) = \left(\frac{1}{3}\left(\frac{3}{2}\right)(1+s)^{\frac{1}{2}} \cdot 1, \frac{1}{3}\left(\frac{3}{2}\right)(1-s)^{\frac{1}{2}} \cdot (-1), \frac{1}{\sqrt{2}}\right)$

$r'(s) = \left(\frac{1}{2}\sqrt{1+s}, -\frac{1}{2}\sqrt{1-s}, \frac{1}{\sqrt{2}}\right)$

$\|r'(s)\| = \sqrt{\frac{1}{4}(1+s) + \frac{1}{4}(1-s) + \frac{1}{2}}$

$= \sqrt{\frac{1}{4} + \frac{1}{4}s + \frac{1}{4} - \frac{1}{4}s + \frac{1}{2}}$

(3/3) $= \sqrt{1} = 1$

b) $\kappa(s) = \frac{\|r'(s) \times r''(s)\|}{\|r'(s)\|^3}$

(3/4) $r''(s) = \left(\frac{1}{2}\left(\frac{1}{2}\right)(1+s)^{-\frac{1}{2}}, -\frac{1}{2}\cdot\frac{1}{2}(1-s)^{-\frac{1}{2}}(-1), 0\right)$

$= \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right)$

$r''(0) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$

$r'(0) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

$\begin{pmatrix} i & j & k \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{4} & 0 \end{pmatrix} i - \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{4} & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

$= -\frac{1}{4\sqrt{2}} i - \frac{1}{4\sqrt{2}} j + \left(\frac{1}{8} + \frac{1}{8}\right) k$

$= \left(\frac{1}{4\sqrt{2}}, -\frac{1}{4\sqrt{2}}, \frac{1}{4}\right)$

$\frac{1}{32} + \frac{1}{32} + \frac{1}{16} = \frac{2}{32} + \frac{1}{16} = \frac{2}{16}$

$\|\left(\frac{1}{4\sqrt{2}}, -\frac{1}{4\sqrt{2}}, \frac{1}{4}\right)\| = \sqrt{\frac{1}{16(2)} + \frac{1}{16(2)} + \frac{1}{16}} = \sqrt{\frac{2}{16}}$

$\boxed{\frac{\sqrt{2}}{4}}$

$\kappa(s) = \frac{\frac{\sqrt{2}}{4}}{1} = \frac{\sqrt{2}}{4}$ ✓

What is $\vec{N}$?

Problem 2.

Recall that one parametrization of the cycloid, the path traced by a point on the edge of a wheel of radius one as the wheel rolls forward, is $\mathbf{r}(t) = (t - \sin t, 1 - \cos t)$.

(a) [5pts.] Find the arclength of $\mathbf{r}(t)$ along the interval $0 \le t \le 2\pi$, that is, as the wheel rolls through one full circle. [You may find it helpful to recall the following half-angle identity: $1 - \cos t = 2\sin^2(\frac{t}{2})$.]

(b) [5pts.] At what times t ~~the curve~~ is the point on the edge of the wheel whose motion is parametrized by this cycloid moving with speed 1?

a) $s = \int_0^{2\pi} \|r'(t)\| \, dt$

$r'(t) = (1 - \cos t, \ 1 + \sin t)$

$\|r'(t)\| = \sqrt{(1-\cos t)^2 + (1+\sin t)^2}$

$= \sqrt{(2\sin^2(\frac{t}{2}))^2 + 1 - 2\sin t + \sin^2 t}$

$= \sqrt{4\sin^4(\frac{t}{2}) + 1 + 2\sin t + \sin^2 t}$

$= \int_0^{2\pi} \sqrt{4\sin^4(\frac{t}{2}) + 1 + 2\sin t + \sin^2 t} \, dt$

$(1-\cos t)(1-\cos t)$

$= 1 - 2\cos t + \cos t^2$

$= \sqrt{1 - 2\cos t + \cos^2 t + 1 + 2\sin t + \sin^2 t}$

$= \sqrt{2 - 2\cos t + 2\sin t + \cos^2 t + \sin^2 t}$

$= \sqrt{2(1 - \cos t + \sin t) + 1}$

(b) 0

Problem 4.

For each of the limits below, either compute the limit or prove that it does not exist. Justify your answers carefully.

(a) [3pts.] $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

(b) [3pts.] $\lim_{(x,y)\to(0,0)} y\tan^{-1}(\frac{1}{x^2})$.

(c) [4pts.] $\lim_{(x,y)\to(0,0)} xy^2\cos(e^{\frac{1}{x^2+50y^2}})$.

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set $y = mx$

$\lim_{x\to 0} \frac{x(mx)}{\sqrt{x^2+(mx)^2}} = \lim_{x\to 0} \frac{x^2 m}{\sqrt{x^2(1+m^2)}}$

$= \lim_{x\to 0} \frac{x^2 m}{x\sqrt{1+m^2}}$

$\lim_{x\to 0} \frac{xm}{\sqrt{1+m^2}} = 0$

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}$

$\frac{r^2\cos\theta\sin\theta}{r}$

$\lim_{r\to 0} r\cos\theta\sin\theta = 0$ ✓

$\boxed{0}$

4 c) $\lim_{(x,y)\to(0,0)} xy^2\cos(e^{\frac{1}{x^2+50y^2}})$

This term will range between -1 and 1

$-|xy^2| \le |xy^2\cos(e^{\frac{1}{x^2+50y^2}})| \le |xy^2|$

Since $\lim_{(x,y)\to(0,0)} xy^2$ is 0, by the squeeze theorem $\lim_{(x,y)\to(0,0)} xy^2\cos(e \ldots)$ is also 0.

~~b) 0~~

$5xe^{5}$

Problem 5.

(a) [5pts.] Compute the first partial derivatives of $g(x, y, z) = xye^{z-x}$.

(b) [5pts.] Either give an example of a function $f(x, y)$ with partial derivatives $f_x(x, y) = 3y^2 - \sin x$ and $f_y(x, y) = 6y + \cos x$, or explain why one cannot exist.

a) $g_x(x,y,z) = (y)\left(1e^{z-x} + xe^{z-x} \cdot (-1)\right)$

$= \boxed{ye^{z-x} - xe^{z-x}}$ ✗

$g_y(x,y,z) = (x)(e^{z-x})(1) = \boxed{xe^{z-x}}$ ✓

$g_z(x,y,z) = (x)(y)\, e^{z-x} \cdot 1 = \boxed{xye^{z-x}}$ ✓

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b) $3y^2x + \cos x$ $\qquad$ $3y^2 + y\cos x$

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If $f(x,y) = 3y^2 + y\cos x$,

then $f_y(x,y) = 6y + \cos x$

If $f(x,y) = 3y^2x + \cos x$,

then $f_x(x,y) = 3y^2 - \sin x$

- A single function with these partial derivatives can not exist. In order for a "y^2" term to exist in $f_x(x,y)$, as it does here, a "x" must be present to account for the constant y^2 term remaining in the partial derivative.
- From this perspective, because each partial derivative has a term with the variable that is being held constant existing by itself, the variable that is having its partial derivative being taken must been with ... var... an...