Math 32A, Lecture 1
Multivariable Calculus

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Question	Points	Score
1	10	4
2	10	4
3	10	4
4	10	3
5	10	7
Total:	50	22

MT1 Physics 1A W16

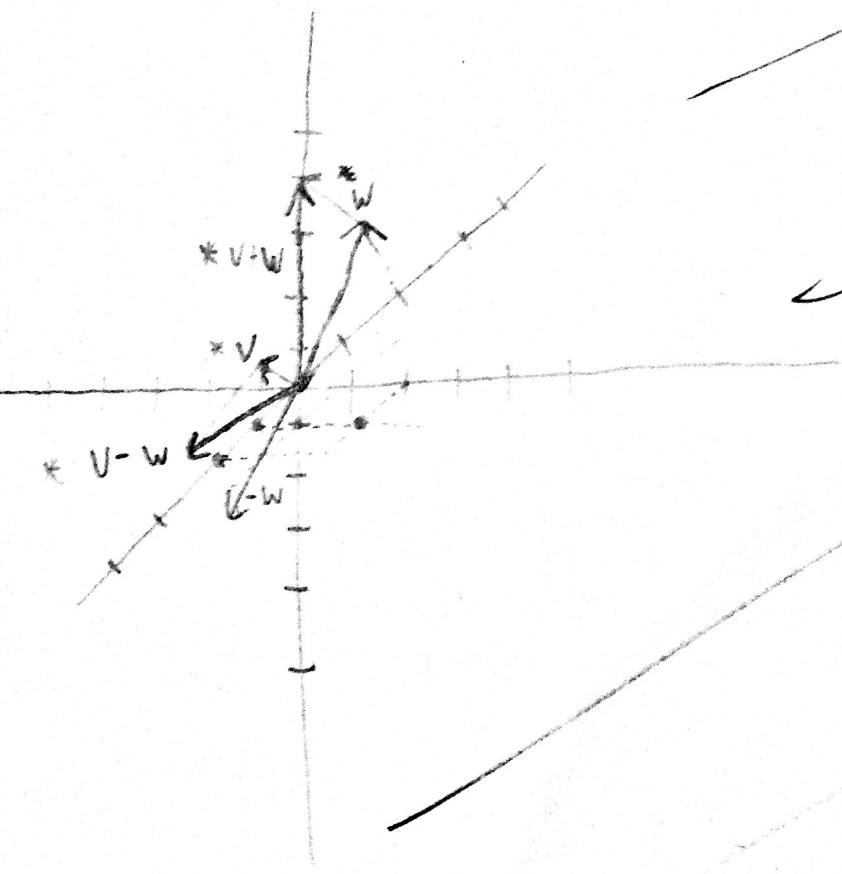
(graph in 3D)

Problem 1.

4 (a) [5pts.] Draw the vectors $\mathbf{v} = \langle 2, -1, 2 \rangle$ and $\mathbf{w} = \langle 1, 2, 4 \rangle$. Sketch $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ on your picture.

0 (b) [5pts.] What is the area of the parallelogram spanned by the unit vectors $\mathbf{e_v}$ and $\mathbf{e_w}$ in the direction of $\mathbf{v}$ and $\mathbf{w}$? [Hint: There is a fast way to do this, using the fact that $\sin^2(\theta) = 1 - \cos^2(\theta)$.]

$e_v = \frac{1}{||v||} v$



b) $v \times u = -u \times v$

$u \cdot v = 0$ if orthogonal

Problem 2.

(a) [5pts.] Let **u** and **v** be two vectors in $\mathbb{R}^3$. Suppose that $\mathbf{u} = \mathbf{u}_{\parallel \mathbf{v}} + \mathbf{u}_{\perp \mathbf{v}}$, where $\mathbf{u}_{\parallel \mathbf{v}}$ is the projection of **u** to **v** and $\mathbf{u}_{\perp \mathbf{v}} = \mathbf{u} - \mathbf{u}_{\parallel \mathbf{v}}$ is orthogonal to **v**. Use the properties of the cross product to prove that $\mathbf{v} \times \mathbf{u} = \mathbf{v} \times \mathbf{u}_{\perp \mathbf{v}}$.

(b) [5pts.] Find the projection of $\mathbf{u} = \langle 1, 3, 2 \rangle$ to $\mathbf{v} = \langle 0, 9, 6 \rangle$, and use part (a) to quickly compute $\mathbf{u} \times \mathbf{v}$.

a)

$v \times u = v \times u_{\perp v}$

$u = u_{\parallel v} + u_{\perp v}$

5/5

$u_{\perp v} = u - u_{\parallel v}$

$v \times (u_{\parallel v} + u_{\perp v}) = v \times u_{\perp v}$

$v \times u_{\parallel v} + v \times u_{\perp v} = v \times u_{\perp v}$

$\cancel{v \times u_{\parallel v} + v \times (u - u_{\parallel v}) = v \times u_{\perp v}}$

$\cancel{v \times u_{\parallel v} + (v \times u - v \times u_{\parallel v}) = v \times u_{\perp v}}$

$v \times u_{\parallel v} + v \times u_{\perp v} = v \times u_{\perp v}$

$u_{\parallel v}$ is the projection of u to v

$u_{\parallel v}$ is a scalar multiple v and their cross product is zero.

$0 + v \times u_{\perp v} = v \times u_{\perp v}$

$v \times u_{\perp v} = v \times u_{\perp v}$ ✓

b)

$$\begin{matrix} i & j & k \\ 0 & 9 & 6 \\ 1 & 3 & 2 \end{matrix}$$

$$\begin{pmatrix} 9 & 6 \\ 3 & 2 \end{pmatrix} i - \begin{pmatrix} 0 & 6 \\ 1 & 2 \end{pmatrix} j + \begin{pmatrix} 0 & 9 \\ 1 & 3 \end{pmatrix} k$$

not how you were asked to compute this.

$\boxed{u_{\parallel v} = \langle 2, 2, \tfrac{4}{3} \rangle} = (18-18)i - (0-6)j + (0-9)k$

$0i + 6j - 9k = v \times u$

weird but incorrect.

$\langle 0, 9, 6 \rangle \times u_{\perp v} = \langle 0, 6, -9 \rangle$

4/5

$$\begin{pmatrix} i & j & k \\ 0 & 9 & 6 \\ x & y & z \end{pmatrix} \rightarrow \begin{pmatrix} 9 & 6 \\ y & z \end{pmatrix} i - \begin{pmatrix} 0 & 6 \\ x & z \end{pmatrix} j + \begin{pmatrix} 0 & 9 \\ x & y \end{pmatrix} k$$

$(9z - 6y)i - (6x)j + (9x)k = \langle 0, 6, -9 \rangle$

$x = -1$

$9z - 6y = 0$

$9z = 6y$

$z = 3$

$u_{\perp v} = \langle -1, \ldots$

$z = y = 0$ is the relevant solution

$u = u_{\parallel v} + u_{\perp v}$

$\langle 1, 3, 2 \rangle = u_{\parallel v} + \langle -1, 1, \tfrac{2}{3} \rangle$

$\langle 1, 3, 2 \rangle - \langle -1, 1, \tfrac{2}{3} \rangle = u_{\parallel v}$

$u_{\parallel v} = \langle 2, 2, \tfrac{4}{3} \rangle$

This is not what I asked for, but is a pretty clever attempt at something else.

Where did that parametrization come from!

$x = 5\cos t + 1$

$y = 5\sin t + 4$

Problem 3.

Let $P = (0,1,4)$, $Q = (2,3,1)$, and $R = (3,-1,-2)$.

Small Q: Did it need to be PQ and PR?

(a) [5pts.] Find an equation for the plane containing P, Q, and R.

(b) [5pts.] Give a parametrization of the intersection of the plane from part (a) with the cylinder $(x-1)^2 + (y-4)^2 = 25$.

a)

4

$\overrightarrow{PQ} \to \langle 2-0, 3-1, 1-4 \rangle$

$\langle 2, 2, -3 \rangle$

$\overrightarrow{QR} \to \langle 3-2, -1-3, -2-1 \rangle$

$\langle 1, -4, -3 \rangle$

$$\begin{vmatrix} i & j & k \\ 2 & 2 & -3 \\ 1 & -4 & -3 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -3 \\ -4 & -3 \end{vmatrix} i - \begin{vmatrix} 2 & -3 \\ 1 & -3 \end{vmatrix} j + \begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} k$$

$(-6-12)i - (-6+3)j + (-8-2)k$

$-18i - (-3)j + (-10)k$

$-18i + 6j - 10k$

$-18x + 6y - 10z = d$

$-18(0) + 6(1) - 10(4) = d$

$6 - 40 = d$

$\boxed{\text{a) } 18x - 6y + 10z = 34}$

b) $18x - 6y + 10z = 34$

$(x-1)^2 + (y-4)^2 = 25$

0

Problem 4.

Consider the following four vector equations for lines.

$$\mathbf{r}_1(t) = \langle 1, 2, 1\rangle + t\langle -3, 3, 6\rangle$$
$$\mathbf{r}_2(s) = \langle 7, 4, 3\rangle + s\langle 2, -2, -4\rangle$$
$$\mathbf{r}_2(u) = \langle 1, 2, 1\rangle + u\langle 4, 3, 2\rangle$$
$$\mathbf{r}_4(v) = \langle 4, -1, -5\rangle + v\langle .5, -.5, -1\rangle$$

(a) [5pts.] Determine which two equations above parametrize the same line.

(b) [5pts.] Find the point of intersection between the line parametrized by $\mathbf{r}_1(t)$ and the line parametrized by $\mathbf{r}_5(w) = \langle 4, 5, 1\rangle + w\langle 1, 0, -1\rangle$, and find the angle between the lines at the point of intersection.

a)

~~$r_1(t) = (1,2,1) + t(3,3,6)$~~

$r_2(s) = (7,4,3) + s(2,-2,-4)$

$r_3(u) = (1,2,1) + u(4,3,2)$

1

let $s = u = 1$

$(7,4,3) + (2,-2,-4) = (1,2,1) + (4,3,2)$

9

b) $1 - 3t = 4 + w$ →

$2 + 3t = 5$

$3t = 3$

$t = 1$

$1 - 3 = 4 + w$

$-2 = 4 + w$

$w = -6$

$r_5(-6) = (4,5,1) + -6(1,0,-1)$

$(4-6, 5, 1+6)$

$\boxed{\text{b) } (-2, 5, 7) \text{ is the point of Intersection}}$ ✓

no angle

2

Problem 5.

Consider the vector-valued function $\mathbf{r}(t) = \langle t, t^2, \sin(\pi t)\rangle$.

(a) [5pts.] Draw the projections of $\mathbf{r}(t)$ to the three coordinate planes, and use these to give a sketch of the space curve determined by $\mathbf{r}(t)$.

(b) [5pts.] Find the equation of the tangent line to $\mathbf{r}(t)$ at $t = \frac{1}{2}$.

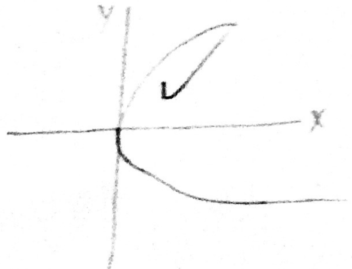
2

a) $x = t$ $y = t^2$

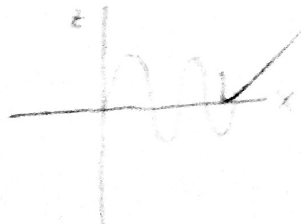
Let $t = 1$

$y = x$

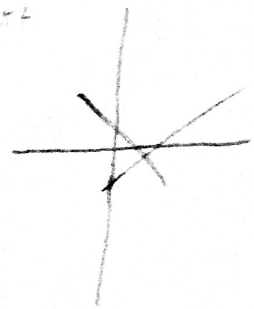
XY Plane



XZ Plane



YZ Plane



space curve ?

Slope of Tangent Line

$r(t) = \langle t, t^2, \sin(\pi t)\rangle$

$r'(t) = \langle 1, 2t, \pi\cos(\pi t)\rangle$

$r'(\frac{1}{2}) = \langle 1, 2(\frac{1}{2}), \pi\cos(\pi \frac{1}{2})\rangle$

$= \langle 1, 1, \pi(0)\rangle$

$= \langle 1, 1, 0\rangle$ Direction vector

Point: $r(\frac{1}{2}) = \langle \frac{1}{2}, \frac{1}{4}, \sin(\frac{\pi}{2})\rangle$

$= \langle \frac{1}{2}, \frac{1}{4}, 1\rangle$

$\boxed{b)\ \langle \frac{1}{2}, \frac{1}{4}, 1\rangle + V\langle 1, 1, 0\rangle}$