1. (a) (3 points) Let $f(x, y) = x^2 + y^2$. Compute the partial derivative f_{xx}.

3

$f_x = 2x$

$f_{xx} = 2$

(b) (5 points) Let $f(u, v, w, x, y, z) = u^2/v + vxyz + \sin(xwv)$. Compute the partial derivative f_{xvz}.

5

$\frac{d}{dx}\left(\frac{d}{dv}\left(\frac{df}{dz}\right)\right)$

$\frac{df}{dz} = 0 + vxy + 0 = vxy$

$\frac{d}{dv}(vxy) = xy$

$\frac{d}{dx}(xy) = y$

(c) (5 points) Compute the following limit:

5

$$\lim_{(x,y)\to(0,2)} (1+x)^{y/x}. = L$$

$\ln\left(\lim_{(x,y)\to(0,2)} (1+x)^{y/x}\right) = \ln L$

$= \lim_{(x,y)\to(0,2)} \ln(1+x)^{y/x} = \lim_{(x,y)\to(0,2)} \frac{y}{x}\ln(1+x)$

$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \frac{0}{0}$. Apply L'hopitals

$\hookrightarrow \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$

$= \lim_{(x,y)\to(0,2)} y \cdot \frac{\ln(1+x)}{x} = 2 \cdot 1$

$= 2$

2. (10 points) Let $f(x, y) = x^2y^3$. Compute the gradient $\nabla f(x, y)$. Then, find the tangent plane to the surface $z = f(x, y)$ at the point $(a, b) = (2, 3)$.

10

$\nabla f(x,y) = (2xy^3, 3x^2y^2)$

$f(2,3) = 2^2(3)^3 = (4)(27) = 108$

$\nabla f(2,3) = (2(2)(3)^3, 3(2)^2(3)^2)$

$= (108, 108)$

Tang. plane @ $(a,b) = (2,3)$

$\hookrightarrow L(a,b) = f(a,b) + ((x,y)-(a,b)) \cdot \nabla f(a,b)$

$L(2,3) = 108 + ((x,y)-(2,3)) \cdot (108, 108)$

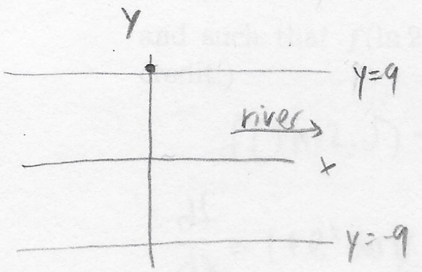
$= 108 + (x-2, y-3) \cdot (108, 108)$

$= 108 + 108x - 216 + 108y - 324$

$z = L(x,y) = 108x + 108y - 432$

$108x + 108y - z = 432$

3. (10 points) Suppose your initial position in the plane is $(0, 9)$. Between the lines $y = 9$ and $y = -9$ is a river. The river's speed at the point (x, y) is $81 - y^2$, where the river runs in the direction of the positive x-axis. Suppose you are in a boat which begins with a constant speed of 1, in the negative y-direction. (So, the velocity of the boat in the y-direction will always be -1.) What will be your position when you reach the bottom of the river? That is, what is your position when you reach the line $y = -9$?



speed @ (x,y) is $81-y^2$

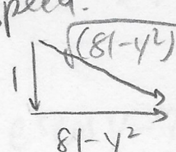
$\|r'(t)\| = 1$ $\downarrow$ -y-dir.

(0, 8)
speed = 81-64 = 17.

$(0,9)$

$(x,-9)$

speed. Travel



$\|r'(t)\| = 81-y^2.$

$t=18.$ $\|r'(t)\| = 81-y^2$ in x-direction.

$r'(t) = (81-y^2, -1)$

$$2\int_0^9 (81-y^2)\,dy$$

$$= 2\left(81y - \frac{y^3}{3}\Big|_0^9\right) = 2\left[\left(81(9) - \frac{729}{3}\right) - (0)\right]$$

$$= 2(729 - 243) = 486 \cdot 2 = 972$$

$81-y^2$

$(972, -9)$

4. (10 points) Find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 1 + e^x \cos y,$$

$$\frac{\partial f}{\partial y} = 14y - e^x \sin y,$$

and such that $f(\ln 2, 0) = \ln 2$. (As usual, you must show your work to receive full credit.)

$f(\ln 2, 0) = \ln 2$

$\frac{\partial f}{\partial x} = 1 + e^x \cos y \Rightarrow x + e^x \cos y \qquad \frac{\partial f}{\partial y} = 7y^2 + e^x \cos y$

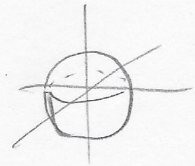
$f(x,y) = x + 7y^2 + e^x \cos y - 2$

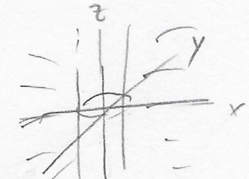
check: $\frac{\partial f}{\partial x} = 1 + e^x \cos y, \quad \frac{\partial f}{\partial y} = 14y + e^x(-\sin y)$

$f(\ln 2, 0) = \ln 2 + 7(0)^2 + e^{\ln 2} \cos 0 - 2$

$= \ln 2 + 2 - 2 = \ln 2$

$f(x,y) = x + 7y^2 + e^x \cos y - 2$





Hyperboloid

5. (15 points) Let D be the solid region in Euclidean space $\mathbf{R}^3$ defined as the set of all (x, y, z) such that $x^2 + y^2 + z^2 \leq 4$, $x^2 + y^2 - 3z^2 \geq 0$ and $x^2 + y^2 \geq 1$. Note that D is a solid region, so its boundary is a surface. Let E denote the boundary surface of D. (If the solid region D were dipped in paint, then the boundary of D is the outer part of D that is covered in paint.)

Let B be the region in Euclidean space $\mathbf{R}^3$ defined as the set of all (x, y, z) such that $y = x$, $x \geq 0$ and $y \geq 0$. Then E and B are surfaces.

Parametrize the intersection of E and B. (Make sure to parametrize the entire intersection. You **MUST** specify the domain of your parameter for any parametrization you give.) (Any parametrization that you write **MUST** use z as a parameter. That is, any parametrization you write must be of the form $r(z) = (x(z), y(z), z)$, where x and y are both functions of z.)

$x^2+y^2-3z^2 \geq 0$

$x^2+y^2+z^2 \leq 4$

$x^2+y^2 \geq 1.$

$x^2+y^2 \leq 4-z^2$

$x^2+y^2 \geq 3z^2$

$3z^2 \leq x^2+y^2 \leq 4-z^2$

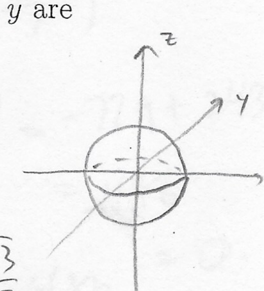
$4z^2 \leq x^2+y^2+z^2 \leq 4$

$x^2+y^2 \geq 1$

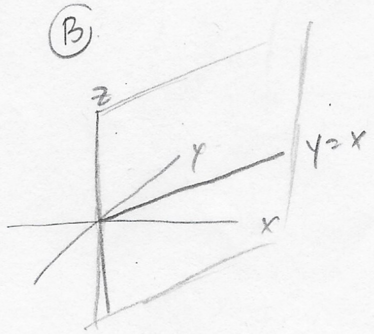
$x^2+y^2 \geq 3z^2$

$3z^2 = 1$

$z = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$



Boundary: $x^2+y^2+z^2=4$



$x^2+y^2+z^2=4.$ $\quad$ $y=x$, $x\geq 0$, $y \geq 0$.

if $y=x$

$2x^2+z^2=4$

$2x^2=4-z^2$ $\quad$ $\sqrt{x^2} = \sqrt{\frac{4-z^2}{2}}$ $\quad$ $x = \pm\sqrt{\frac{4-z^2}{2}}$

N=1

Intersection:

$r(z) = \left(\sqrt{\frac{4-z^2}{2}}, \sqrt{\frac{4-z^2}{2}}, z\right)$, for $\cancel{-2} \leq z \leq 2$

~~$r(z) = \left(-\sqrt{\frac{4-z^2}{2}}, -\sqrt{\frac{4-z^2}{2}}, z\right)$, for $-z \leq z \leq 2$~~