Problem 1: Consider the surface $x^2 + (y-2)^3 - z = 1$

(a) Parameterize the curve obtained by taking the trace of this surface at $y = 0$.

(b) Use your parameterization to find the unit tangent vector and unit normal vector to the curve of part (a) at $x = -1$.

(c) Sketch the curve of part (a) in the xz-plane. Label your axes clearly and mark the coordinates of the point at which the curve intersects the z-axis. Sketch the unit tangent and normal vectors at $x = -1$.

Solution:

(a) The trace $y = 0$ is obtained by putting $y = 0$ in the equation. We get the equation $x^2 - 8 - z + 1$ or $z = x^2 - 9$. This can be parameterized by simply putting $t = x$. So we get a parameterization

$$\mathbf{r}(t) = \langle t, 0, t^2 - 9\rangle, -\infty < t < \infty.$$

(b) To find the unit tangent vector, we first find $\mathbf{r}'(t)$.

$$\mathbf{r}'(t) = \langle 1, 0, 2t\rangle.$$

At $x = -1$, $t = -1$, so $\mathbf{r}'(-1) = \langle 1, 0, -2$. To get the unit tangent vector simply divide by the magnitude. Hence,

$$\mathbf{T}(-1) = \frac{1}{\sqrt{5}}\langle 1, 0, -2\rangle.$$

Since this vector is in the xz-plane and the normal vector also has to be in the xz-plane,

$$\mathbf{N} = \pm\frac{1}{\sqrt{5}}\langle 2, 0, 1\rangle$$

as we can simply exchange the coordinates and negate one of them. To figure out the sign, we compute $\frac{dz}{dx}, \frac{d^2z}{dx^2}$ at $x = -1$.

$$\frac{dz}{dx} = 2x = -2 < 0, \ \frac{d^2z}{dx^2} = 2 > 0.$$

Hence, z is decreasing and concave upwards, so the normal vector points in the positive x-direction and positive z-direction. Hence,

$$\mathbf{N} = \frac{1}{\sqrt{5}}\langle 2, 0, 1\rangle.$$

Problem 2:

(a) Compute $f_{yzy}(x, y, z)$ where

$$f(x, y, z) = (y^2 + z^2)e^{xy} + e^{x \sin y}(\sin x)^{\cos(yx)}.$$

(b) Show that there is no function $f(x, y)$ such that

$$f_x(x, y) = 2xy + \sin y, \ f_y(x, y) = x^2 + \sin y.$$

Solution:

(a) Using Clairaut's theorem, $f_{yzy} = f_{zyy}$. The last term in the sum doesn't have z at all. So

$$f_z = 2ze^{xy},\ f_zy = 2zxe^{xy},\ f_zyy = 2zx^2e^{xy}.$$

(b) There are two ways to solve this problem. The easier method is to note that

$$f_xy = \frac{\partial f_x}{\partial y} = 2x + \cos y$$

whereas

$$f_yx = \frac{\partial f_y}{\partial x} = 2x.$$

Since both are continuous functions, Clairaut's theorem applies and requires that $f_yx = f_xy$. Since this is not the case, no f can exist with these partial derivatives.

The other way is to integrate the partial derivatives. However, when doing so, it is important to remember that when integrating f_x with respect to x we can add any function of y and still get a valid f and when integrating f_y with respect to y, we can add any function of x. So, we get

$$\int f_x\,dx = x^2y + x\sin(y) + g(y) = x^2y - \cos(y) + h(x) = \int f_y\,dy.$$

We can make $g(y) = -\cos y$ to cancel out that term, but there is no way to find an $h(x)$ that will give $x\sin(y)$ as that is a function that involves both x and y. Hence, f cannot exist.

Problem 3: Consider the space curve C parameterized by

$$\mathbf{r}(u) = \langle \cos(e^u), \sin(e^u), e^u \rangle, \quad -\infty < u < \infty$$

(a) Find an arc length parameterization for C.

(b) Compute the curvature of C at the point $(\cos 1, \sin 1, 1)$.

(c) Suppose an object moves along the curve C at a constant speed of $10ms^{-1}$. Find the acceleration vector $\mathbf{a}$ at the point $(\cos 1, \sin 1, 1)$.

Solution:

(a) Let us compute the arc length function using $\mathbf{r}(0)$ as the start point (any start point is ok for this problem).

$$\mathbf{r}'(u) = e^u\langle -\sin(e^u), \cos(e^u), 1\rangle$$

and hence

$$\|\mathbf{r}'(u)\| = \sqrt{2}e^u.$$

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du = \sqrt{2}\int_0^t e^u\,du = \sqrt{2}(e^t - 1).$$

Hence,

$$t = \ln\left(\frac{s}{\sqrt{2}} + 1\right).$$

Plugging in this in the parameterization and cancelling the exponential with the logarithm, we have

$$\mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}+1\right), \sin\left(\frac{s}{\sqrt{2}}+1\right), \left(\frac{s}{\sqrt{2}}+1\right)\right\rangle.$$

(b) To compute the curvature, we can simply use the original parameterization, find the unit tangent vector, differentiate it and divide by the speed with respect to t.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2}}\langle -\sin(e^u), \cos(e^u), 1\rangle.$$

Hence,

$$\mathbf{T}'(t) = \frac{e^u}{\sqrt{2}}\langle -\cos(e^u), -\sin(e^u), 0\rangle.$$

Thus,

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{2}.$$

At the point we are looking at $t = 1$, but since the curvature is constant, that doesn't really matter.

(c) There are two ways to solve this. One is to find T, N, κ and then use the formulas for tangential and normal acceleration. An easier way, however, is to note that if we are moving at constant speed 10, then $s = 10t$, where here s is arc length and t is time. Hence, parameterizing using time, we have

$$\mathbf{r}(t) = \left\langle \cos\left(\frac{10t}{\sqrt{2}}+1\right), \sin\left(\frac{10t}{\sqrt{2}}+1\right), \left(\frac{10t}{\sqrt{2}}+1\right)\right\rangle.$$

To find acceleration, simply differentiate twice to get

$$\mathbf{r}''(t) = 50\left\langle -\cos\left(\frac{10t}{\sqrt{2}}+1\right), -\sin\left(\frac{10t}{\sqrt{2}}+1\right), 0\right\rangle$$

and then plug in $t = 0$ to get

$$\mathbf{r}''(0) = 50\langle -\sin 1, -\cos 1, 0\rangle.$$

Problem 4: Let $f(x, y)$ be defined as

$$f(x,y) = \begin{cases} \frac{x^4-y^2}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

For the following 2 limits, either compute the limit or show that it does not exist:

(a) $\lim_{(x,y)\to(1,0)} f(x,y)$.

(b) $\lim_{(x,y)\to(0,0)} f(x,y)$.

Solution:

(a) Both numerator and denominator are continuous at $(1, 0)$ so we can simply plug in the values to get the limit equals 1.

(b) If we approach along the line $y = 0$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^4}{x^4} = 1.$$

If we approach along the line $x = 0$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

Since we get different limits along different paths, the limit does not exist.