MATH 32A, FALL 2019, MIDTERM 1 SOLUTIONS

Problem 1: Let P, Q, R be three points in $\mathbb{R}^3$ that do not lie on the same line.

(a) Suppose the area of the triangle with vertices P, Q and R is 2. Compute the magnitude of $\overrightarrow{PQ} \times \overrightarrow{PR}$.

(b) Let S be another point in $\mathbb{R}^3$ such that $||\overrightarrow{PS}|| = 5$. Show that the volume of the parallelepiped spanned by $\overrightarrow{PQ}, \overrightarrow{PR}$ and $\overrightarrow{PS}$ is less than or equal to 20.

Solution:

(a) The area of the parallelogram spanned by $\overrightarrow{PQ}, \overrightarrow{PR}$ is $\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$. The triangle is half of the parallelogram, hence

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = 0.$$

(b) There are two ways to solve this problem. The first is to use the formula for the volume via the triple product and then to remember the magnitude angle formula for dot products.

$$\begin{aligned} V &= \|\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR})\| \\ &= \|\overrightarrow{PS}\|\|\overrightarrow{PQ} \times \overrightarrow{PR}\|| \cos\theta| \\ &\leq \|\overrightarrow{PS}\|\|\overrightarrow{PQ} \times \overrightarrow{PR}\| \\ &= 20 \end{aligned}$$

since $-1 \leq \cos\theta \leq 1$. Here θ is the angle between $\overrightarrow{PS}$ and $\overrightarrow{PQ} \times \overrightarrow{PS}$.

The other way to solve the problem is to note that the volume will be the area of the base parallelogram spanned by $\overrightarrow{PQ}, \overrightarrow{PR}$ times the height. The height is smaller than the length of $\overrightarrow{PS}$ because it forms a right angled triangle with $\overrightarrow{PS}$ the hypotenuse.

Problem 2: Let $P = (1,0,0), Q = (1,1,1), R = (0,1,0), S = (0,0,1)$.

(a) Find the equation of the plane containing P, Q, R.

(b) Show that there is no plane in $\mathbb{R}^3$ that contains all four of P, Q, R and S.

(c) Parameterize the line passing through S that is perpendicular to the plane containing P, Q, R.

(d) Find the intersection of the line from part (c) with the plane from part (a).

Solution:

(a) To find a normal vector to the plane, we can take $\overrightarrow{PQ} \times \overrightarrow{PR}$. $\overrightarrow{PQ} = \langle 0,1,1\rangle$, since we simply subtract the coordinate of P from those of Q. $\overrightarrow{PR} = \langle -1,1,0\rangle$. Hence, $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1,-1,1\rangle$. Since, the components of the normal vector give us the coefficients of the equation for the plane, the plane must have equation

$$-x - y + z = d.$$

To solve for d, simply plug in any of the three points, P, Q, R. Plugging in P, we get $d = -1$. Hence, the equation is

$$-x - y + z = -1.$$

(b) There is only one plane containing P ,Q and R, which has equation given by part (a). We just need to show that S does not satisfy this equation. But for S, $-x - y + z = 1$ and not -1.

(c) S is a point on the line. So to parameterize it, all we need to know is the direction vector. The direction vector will be perpendicular to the plane from part (a), so will be the normal vector to the plane. Hence, we have a direction vector $\langle -1, -1, 1\rangle$ and the parametric equation is

$$\mathbf{r}(t) = \langle 0, 0, 1\rangle + t\langle -1, -1, 1\rangle, -\infty < t < \infty.$$

(d) Let (x, y, z) be the intersection point. Then, for some value of t,

$$x = -t, y = -t, z = 1 + t$$

because the point must lie on the line from part (c). Additionally, the point must also satisfy the equation for the plane from part (a), and so we can solve for t, by plugging in these x, y, z values into the equation. We get

$$t + t + 1 + t = -1 \Rightarrow t = \frac{-2}{3}.$$

Plugging this back into the equations for x, y, z, we get

$$x = \frac{2}{3} = y, z = \frac{1}{3}.$$

Problem 3: Consider the space curve parameterized by the vector valued function

$$\mathbf{r}(t) = \langle 2\cos t, -4\sin t, t^2\rangle, \quad -\infty < t < \infty$$

Find a parametric equation of the tangent line to the curve at the point $\left\langle 0, -4, \frac{\pi^2}{4}\right\rangle$.

Solution: To find the parametric equation for the line we need a point on the line and a direction vector. $\left\langle 0, -4, \frac{\pi^2}{4}\right\rangle$ is a point on the line. So we just need a direction vector. This direction vector will be $\mathbf{r}'(t_0)$ where t_0 is the value of t that gives us $\left\langle 0, -4, \frac{\pi^2}{4}\right\rangle$. So we need to differentiate $\mathbf{r}$ and also find this t-value. Since $t_0^2 = \frac{\pi^2}{4}$, $t_0 = \pm\frac{\pi}{2}$ and looking at the sine values tells us that $t_0 = \frac{\pi}{2}$. Hence,

$$\mathbf{r}'(t_0) = \langle -2\sin t_0, -4\cos t_0, 2t_0\langle = \langle -2, 0, \pi\rangle.$$

Thus, the line is parameterized as

$$\left\langle 0, -4, \frac{\pi^2}{4}\right\rangle + s\langle -2, 0, \pi\rangle, -\infty < s\infty.$$

Problem 4: An object is launched from the ground at an angle of $60°$ at a speed of $10ms^{-1}$. Due to wind in the horizontal direction, its velocity vector is given by

$$\mathbf{r}'(t) = (3\mathbf{i} + (5\sqrt{3} - 10t)\mathbf{k})ms^{-1},$$

where the x-axis is the ground and the y-axis is the vertical direction.

(a) Describe the position vector $\mathbf{r}(t)$ of the object as a function of time, if $\mathbf{r}(0) = \langle 0, 0 \rangle$.

(b) Find the time t at which the object hits the ground and compute the horizontal distance of the point of impact from the starting point of the object.

Solution:

(a) To get $\mathbf{r}(t)$ we integrate $\mathbf{r}'(t)$.

$$\mathbf{r}(t) = \mathbf{r}(t) - \mathbf{r}(0) = \int_0^t \mathbf{r}'(u)\, du = \langle 3t, 0, 5\sqrt{3}t - 5t^2 \langle m.$$

(b) When the object hits the ground, its z-coordinate must be 0. Solving for t, we get

$$5\sqrt{3}t = 5t^2$$

so since $t = 0$ is just the starting time, we have the only possible t value$t = \sqrt{3}s$. To find the horizontal distance, note that the initial x-coordinate is 0, and hence the horizontal distance is just the x-coordtinate at the time of impact, which is $3t = 3\sqrt{3}m$.