Math 32A, Winger 2020
Multivariable Calculus

Final Exam

Instructions: You have 3 hours to complete this exam. For full credit, show all work and justify your answers. **Box your answers.** Write your solutions in the space below the questions. Write your name and UID clearly in the space below

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Question	Points	Score
1	8	
2	15	
3	12	
4	16	
5	12	
6	10	
7	8	
8	15	
9	1	
Total:	97	

Score is computed out of 96. The last question is bonus.

Problem 1. *8pts.*

Write 'T' for true or 'F' for false. (You do not need to show any work for this question.)

(a) F If u, v, w are three dimensional vectors and w is non-zero and $u \cdot w = 0$ and $v \cdot w = 0$, then u and v are parallel.

(b) T For a vector valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, we always have $\|\mathbf{r}''(t)\| \geq |x''(t)|$.

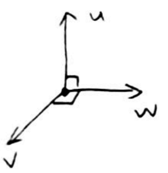
(c) F For any vector valued function $\mathbf{r}(t)$, we have

$$\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{r}(t)) = 2\mathbf{r}'(t) \times \mathbf{r}(t).$$

(d) F For a function f in two variables, if $\lim_{x \to 0} f(x, 0) = 0$ and $\lim_{y \to 0} f(0, y) = 0$, then $\lim_{(x,y) \to (0,0)} f(x, y) = 0$.

(e) T Let f be a function in two variables, and P be a critical point. Suppose that $f_{xx}, f_{xy}, f_{yy}, f_{yx}$ are continuous near P. (This means they are continuous at P, and also there is a disk around P on which they are continuous.) Then if $f_{xx}(P) f_{yy}(P) - (f_{xy}(P))^2 > 0$ then P is either a local minimum or a local maximum.

$$\frac{d}{dt}(r(t) \times r(t)) = [r'(t) \times r(t)] + [r(t) \times r'(t)]$$



Problem 2.

Write the letter representing the correct answer in the box on the left. (You do not need to show any work for this question.)

(a) [3pts.] The vector valued function $\mathbf{r}(t) = \langle 5 + \sin t, 2, 4 - \cos(t)\rangle$ traces a circle centred at a point (a, b, c) with radius r. Find $a + b + c + r$.

A. 3 B. 6 C. 10 D. 11 E. 12

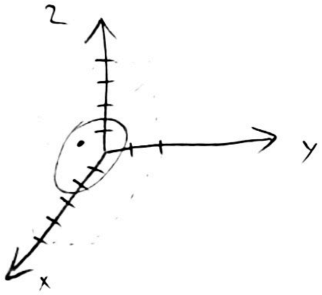
cente = (5,2,4)

radius = 1

$5+2+4+1 = 12$

(b) [3pts.] The vector valued function $\mathbf{r}(t) = \langle 5+\sin t, 2, 4-\cos(t)\rangle$ traces a circle. This circle is parallel to which of the following planes.

A. $x + z = 0$ B. xy plane C. yz plane D. xz plane E. None of the above



(c) [3pts.] Given a particle traversing the path $\mathbf{r}(t) = \langle t, \sin(t), t^2 + 2t\rangle$. Find the speed of the particle at $t = 0$.

A. 1 B. $\sqrt{2}$ C. 2 D. -2 E. $\sqrt{6}$

$\vec{r}(t) = \langle t, \sin(t), t^2+2t\rangle$

$\vec{r}'(t) : \langle 1, \cos(t), 2t+2\rangle$

$\vec{r}'(0) = \langle 1, 1, 2\rangle$

speed $= \|\vec{r}'(t)\| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$

(d) [3pts.] Find the curvature of $y = x^2 + \cos(x)$ at $(0, 1)$.

A. 1 B. 2 C. $\frac{2}{2^{3/2}}$ D. $\frac{2}{5^{3/2}}$ E. 0 F. $\sqrt{6}$

$f(x) = x^2 + \cos x$

$f'(x) = 2x - \sin x \quad f'(0) = 0$

$f''(x) = 2 - \cos x \quad f''(0) = 1$

$K = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$

$= \frac{1}{(1+0)^{3/2}} = 1$

(e) [3pts.] Find the projection of $\langle 2, 3\rangle$ along the vector $\langle 1, -1\rangle$.

A. 0 B. -1 C. $-\frac{1}{\sqrt{2}}$ D. $\langle \frac{1}{2}, -\frac{1}{2}\rangle$ E. $\langle -\frac{1}{2}, \frac{1}{2}\rangle$ F. $\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle$

$\vec{u} = \langle 2, 3\rangle \quad \vec{v} = \langle 1, -1\rangle$

$\sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\vec{u}_{\parallel \vec{v}} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$

$= \left(\frac{2-3}{2}\right)\langle 1, -1\rangle$

$= \left(-\frac{1}{2}\right)\langle 1, -1\rangle$

$= \langle -\frac{1}{2}, \frac{1}{2}\rangle$

Problem 3.

For each function $f(x, y)$ and value c, choose which of the pictures on the last page of this exam $A, B, C, D, E, F, G, H, I, K$ best matches the $f(x, y) = c$ contour. (The figures are at the end, so it's easier for you to remove the page to look at.)

(a) [2pts.] $f(x, y) = 2^{3(x-y)}$, $c = 1$.

$3(x-y)=0$

$x-y=0$

$x=y$

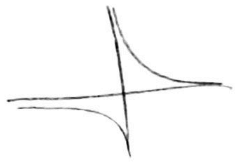
$\boxed{H}$

(b) [2pts.] $f(x, y) = (5xy - 1)^2$, $c = 0$.

$5xy-1=0$

$5xy=1$

$y=\frac{1}{5x}$



$\boxed{D}$

(c) [2pts.] $f(x, y) = 2e^{\sin(2(x^2+y^2)+0.01)}$, $c = 2$.

$e^{\sin(2(x^2+y^2)+0.01)}=1$

$\sin(2(x^2+y^2)+0.01)=0$

$2(x^2+y^2+0.01)=0, \pi,$

$x^2+y^2+0.01=0, \frac{\pi}{2}$

$x^2+y^2=-0.01, \frac{\pi}{2}-0.01, \ldots$

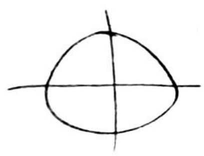
$\boxed{A}$

(d) [2pts.] $f(x, y) = 2^{\sqrt{x^2+y^2}}$, $c = 5$.

$\sqrt{x^2+y^2}\ln 2=\ln 5$

$\sqrt{x^2+y^2}=\frac{\ln 5}{\ln 2}$

$x^2+y^2=\left(\frac{\ln 5}{\ln 2}\right)^2$



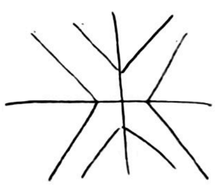
$\boxed{C}$

(e) [2pts.] $f(x, y) = 2^{\sin\left(\frac{\pi}{2}-2|x|+2|y|\right)}$, $c = 1$.

$\sin\left(\frac{\pi}{2}-2|x|+2|y|\right)=0$

$\frac{\pi}{2}-2|x|+2|y|=0, \pi, 2\pi, \ldots$

$-2|x|+2|y|=-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$



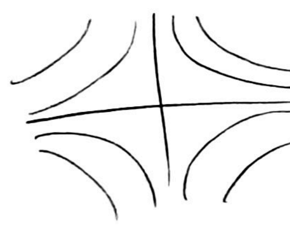
$\boxed{I}$

(f) [2pts.] $f(x, y) = \sin(2xy)^2$, $c = 0$.

$\sin(2xy)=0$

$2xy=0, \pi, 2\pi, \ldots$

$y=0, \frac{\pi}{2x}, \frac{\pi}{x}, \ldots$



$\boxed{E}$

Problem 4.

Consider the function $f(x, y) = x^2 - y^2 - xy + 5x$.

(a) [2pts.] Compute the gradient of f at point $(1, -1)$.

$f_x = 2x - y + 5$

$f_y = -2y - x$

$\nabla f = \langle 2x - y + 5, -2y - x \rangle$

$\nabla f_{(1,-1)} = \langle 2 - (-1) + 5, -2(-1) - 1 \rangle$

$= \boxed{\langle 8, 1 \rangle}$

(b) [3pts.] Find the critical point(s) of f.

$y = 2x + 5$

$x = -2y$

$y = 2(-2y) + 5$

$y = -4y + 5$

$5y = 5$

$y = 1$

$x = -2(1) = -2$

$\boxed{(-2, 1)}$

(c) [3pts.] For each critical point you found, determine whether it is a local minimum, local maximum, or saddle point.

$f_{xx} = 2, \; f_{yy} = -2, \; f_{xy} = -1$

$D(-2, 1) = f_{xx} f_{yy} - (f_{xy})^2$

$= 2(-2) - (-1)^2$

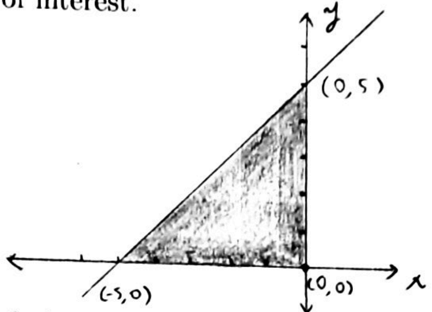
$= -4 - 1$

$= -5$

< 0

$\boxed{(-2, 1): \text{saddlepoint}}$

(d) [2pts.] Draw a picture of the region $x \leq 0$, $y \geq 0$, $y \leq x + 5$. Label the axes and any intercepts of interest.



(e) [6pts.] Find the maximum value of f on each segment of the boundary of the region $x \leq 0$, $y \geq 0$, $y \leq x + 5$.

on $x = 0$

$$f(0,t) = 0^2 - t^2 - 0 + 0$$
$$= -t^2$$

max at $t = 0$, $f(0,0) = 0$

on $y = 0$

$$f(t,0) = t^2 - 0 - 0 + 5t$$
$$= t^2 + 5t$$

max at $t = 0$, $f(0,0) = 0$

$t = -5$, $f(-5,0) = 0$

on $y = x + 5$

$$f(t, t+5) = t^2 - (t+5)^2 - t(t+5) + 5t$$
$$= \cancel{t^2} \cancel{-t^2} - 10t - 25 - t^2 \cancel{-5t} + \cancel{5t}$$
$$= -(t^2 + 10t + 25)$$

max at $t = -5$, $f(-5,0) = 0$

on $x = 0$:
0 at $(0,0)$

on $y = 0$
0 at $(0,0)$ and $(-5,0)$

on $y = x + 5$
0 at $(-5,0)$

Problem 5.

Consider the function $f(x, y) = x^3 + 2xy - 2y^2 - 10x$.

(a) [2pts.] Compute the gradient of f at point $(-2, 1)$.

$$f_x = 3x^2 + 2y - 10$$

$$f_y = 2x - 4y$$

$$\nabla f = \langle 3x^2 + 2x - 10, 2x - 4y \rangle$$

$12 + 2 - 10$ $\quad$ $-4 - 4$

$$\nabla f_{(-2,1)} = \langle 3(-2)^2 + 2(1) + 10, 2(-2) - 4(1) \rangle = \boxed{\langle 4, -8 \rangle}$$

(b) [3pts.] Write down the equation for the tangent plane to the surface this function defines at the point $(-2, 1)$.

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= 6 + 4(x+2) - 8(y-1)$$

$$= 6 + 4x + 8 - 8y + 8$$

$$z = 4x - 8y + 22$$

$$\boxed{4x - 8y - z = -22}$$

$$f(-2,1) = (-2)^3 + 2(-2)(1) - 2(1)^2 - 10(-2)$$

$$= -8 - 4 - 2 + 20 = 6$$

$$f_x = 4$$

$$f_y = -8$$

(c) [5pts.] Find the critical points of f.

$3x^2+2y-10=0$

$2x-4y=0$

$\frac{2x}{4}=\frac{4y}{4}$

$y=\frac{1}{2}x$

$3x^2+x-10=0$

$x=\frac{-1\pm\sqrt{1^2-4(3)(-10)}}{2(3)}=\frac{-1\pm 11}{6}$

$=\frac{-1+11}{6},\ \frac{-1-11}{6}$

$=\frac{10}{6},\ -\frac{12}{6}=5/3,\ -2$

$y=5/6,\ -1$

$\boxed{CP: (5/3, 5/6) \& (-2,-1)}$

(d) [2pts.] Use the second derivative test to determine whether they are local maxima, minima or saddle points.

$f_{xx}=6x$

$f_{yy}=-4$

$f_{xy}=2$

$D(5/3,5/6)=f_{xx}f_{yy}-(f_{xy})^2$

$=6(\frac{5}{3})(-4)-2^2$

$=-40-4=-44<0$

$\boxed{\begin{array}{l}(5/3,5/6)\text{: saddlepoint}\\ (-2,-1)\text{: local max}\end{array}}$

$D(-2,-1)=6(-2)(-4)-2^2$

$=48-4$

$=44>0$

$f_{xx}=6(-2)=-12<0$

Problem 6.

Consider the function $f(x, y, z) = 3x + yz + 3\cos(x + y + z)$.

(a) [2pts.] Compute the gradient of f.

$f_x = 3 - 3\sin(x+y+z)$

$f_y = z - 3\sin(x+y+z)$

$f_z = y - 3\sin(x+y+z)$

$$\boxed{\nabla f = \langle 3-3\sin(x+y+z),\ z-3\sin(x+y+z),\ y-3\sin(x+y+z)\rangle}$$

(b) [2pts.] Find the directional derivative of f in the $\langle 1, 1, 1\rangle$ direction at the point $(2, -1, -1)$.

$D_{\langle 1,1,1\rangle} f(2,-1,-1) = \nabla f_{(2,-1,-1)} \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$\frac{\langle 1,1,1\rangle}{\sqrt{1^2+1^2+1^2}} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$= \langle 3-0, -1-0, -1-0\rangle \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$= \langle 3, -1, -1\rangle \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

$= \frac{3}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}$

$= \boxed{1/\sqrt{3}}$

(c) [4pts.] Find the equation of the tangent plane to the surface $f(x, y, z) = 2$ at the point $(0, 1, -1)$.

$\nabla f(0,1,-1) = \langle 3-0, -1-0, 1-0 \rangle = \langle 3, -1, 1 \rangle$

$3(x-0) - 1(y-1) + 1(z+1) = 0$

$3x - y + 1 + z + 1 = 0$

$\boxed{3x - y + z + 2 = 0}$

(d) [2pts.] Find a point $P = (a, b, c)$ such that for $d = f(a, b. c)$, the tangent plane to $f(x, y, z) = d$ at P is parallel to $x + 2y + 3z = 0$.

$\lambda \langle 1, 2, 3 \rangle = \langle 3 - 3\sin(x+y+z), z - 3\sin(x+y+z), y - 3\sin(x+y+z) \rangle$

$$\begin{cases} 3 - 3\sin(x+y+z) = \lambda \\ z - 3\sin(x+y+z) = 2\lambda \\ y - 3\sin(x+y+z) = 3\lambda \end{cases}$$

let $\lambda = 3$

$3\sin(x+y+z) = 0$ if $\lambda = 3$

$z - 0 = 6 \quad z = 6$

$y - 0 = 9 \quad y = 9$

$3\sin(6+9+x) = 0$

$\sin(6+9+x) = 0$

$x = -15$

$\boxed{P: (-15, 9, 6)}$

Problem 7.

(a) [4pts.] Compute $\frac{\partial g}{\partial s}$ where $g(x, y) = x^2 + y^2$ and $x = \cos(s + t)$, $y = s^2 - t$.

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s}$$

$$= 2x(-\sin(s+t)) + 2y(2s)$$

$$= -2\cos(s+t)\sin(s+t) + 2(s^2-t)(2s)$$

$$= \boxed{-2\cos(s+t)\sin(s+t) + 4s^3 - 4st}$$

(b) [4pts.] Let $f(x, y, z) = x^2y + \sin(yz) + y^2 - 2$. Consider the surface $f(x, y, z) = 0$, and see it as the graph of a function $z = g(x, y)$. Find $\frac{\partial g}{\partial x}$, wherever it is defined.
Alternately, here is the same problem in the notation of the textbook: Find $\frac{\partial z}{\partial x}$ for the surface $x^2y + \sin(yz) + y^2 = 2$.

$F_x = 2xy$

$F_z = y\cos(yz)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy}{y\cos(yz)}$$

$$= \boxed{-\frac{2x}{\cos(yz)}}$$

Problem 8.

Let $\mathbf{u} = \langle 3, 1, -2 \rangle$.

(a) [5pts.] Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v} \times \mathbf{w} = 0$, $\mathbf{u} \cdot \mathbf{v} = 3$, $\mathbf{w} \cdot \mathbf{u} = -2$.

$w = \lambda v$ since $v \times w = 0$

$u \cdot v = 3$

$u \cdot w = u \cdot \lambda v = -2$

$= \lambda (u \cdot v) = -2$

$\lambda = -\frac{2}{3}$

$u \cdot v = 3x + y - 2z = 3$

let $v = \langle 1, 0, 0 \rangle$

$w = -\frac{2}{3} \langle 1, 0, 0 \rangle$

$= \langle -\frac{2}{3}, 0, 0 \rangle$

$$\boxed{\begin{aligned} \vec{v} &= \langle 1, 0, 0 \rangle \\ \vec{w} &= \langle -\tfrac{2}{3}, 0, 0 \rangle \end{aligned}}$$

(b) [10pts.] For $\mathbf{v}$ and $\mathbf{w}$ as in the previous part, that is satisfying $\mathbf{v} \times \mathbf{w} = 0$, $\mathbf{u} \cdot \mathbf{v} = 3$, $\mathbf{w} \cdot \mathbf{u} = -2$, find the maximum possible value of $\mathbf{v} \cdot \mathbf{w}$. (The value you get from doing Lagrange multipliers is a maximum; you don't have to justify this.)

maximize $f(x,y,z) = -\frac{2}{3}(x^2+y^2+z^2)$

$$g(x,y,z) = \langle 3,1,-2\rangle\langle x,y,z\rangle = 3$$
$$= 3x+y-2z = 3$$
$$= 3x+y-2z-3$$

$$\nabla g = \langle 3,1,-2\rangle \neq 0$$

$$\nabla f = \langle -\tfrac{4}{3}x, -\tfrac{4}{3}y, -\tfrac{4}{3}z\rangle = \lambda\langle 3,1,-2\rangle$$

$$\begin{cases} -\frac{4}{3}x = 3\lambda \\ -\frac{4}{3}y = \lambda \\ -\frac{4}{3}z = -2 \\ 3x+y-2z=3 \end{cases}$$

$x = -\frac{9}{4}\lambda = -\frac{9}{4}(-\frac{2}{7}) = \frac{9}{14}$

$y = -\frac{3}{4}\lambda = -\frac{3}{4}(-\frac{2}{7}) = \frac{3}{14}$

$z = \frac{6}{4}\lambda = \frac{\cancel{6}^{3}}{\cancel{4}_{2}}(-\frac{2}{7}) = -\frac{3}{7}$

$$-\frac{27}{4}\lambda - \frac{3}{4}\lambda - \frac{12}{4}\lambda = 3$$
$$-\frac{42}{4}\lambda = 3$$
$$\lambda = \cancel{3}^{1}\left(\frac{\cancel{4}^{2}}{\cancel{42}_{\cancel{14}\,7}}\right)$$
$$= -\frac{2}{7}$$

$$V\cdot W = -\frac{2}{3}\left(\left(\frac{9}{14}\right)^2 + \left(\frac{3}{14}\right)^2 + \left(-\frac{3}{7}\right)^2\right)$$
$$= -\frac{2}{3}\left(\frac{81}{196} + \frac{9}{196} + \frac{9}{49}\right)$$
$$= -\frac{2}{3}\left(\frac{81}{196} + \frac{9}{196} + \frac{36}{196}\right)$$
$$= -\frac{\cancel{2}^{1}}{\cancel{3}_{1}}\left(\frac{\cancel{126}^{\cancel{42}\,\cancel{21}\,3}}{\cancel{196}_{\cancel{98}\,\cancel{49}\,7}}\right) = \boxed{-\frac{3}{7}}$$

Problem 9. *1pts.*

(Bonus) Find, with justification, the maximum value of $\frac{2(a+b)}{a+b+c} + \frac{a+3b}{a+2b+c} - \frac{2a+3b}{a+b+2c}$ for $a, b, c > 0$.

(Note: Collaboration is okay for this problem, but I will not provide help with it.)

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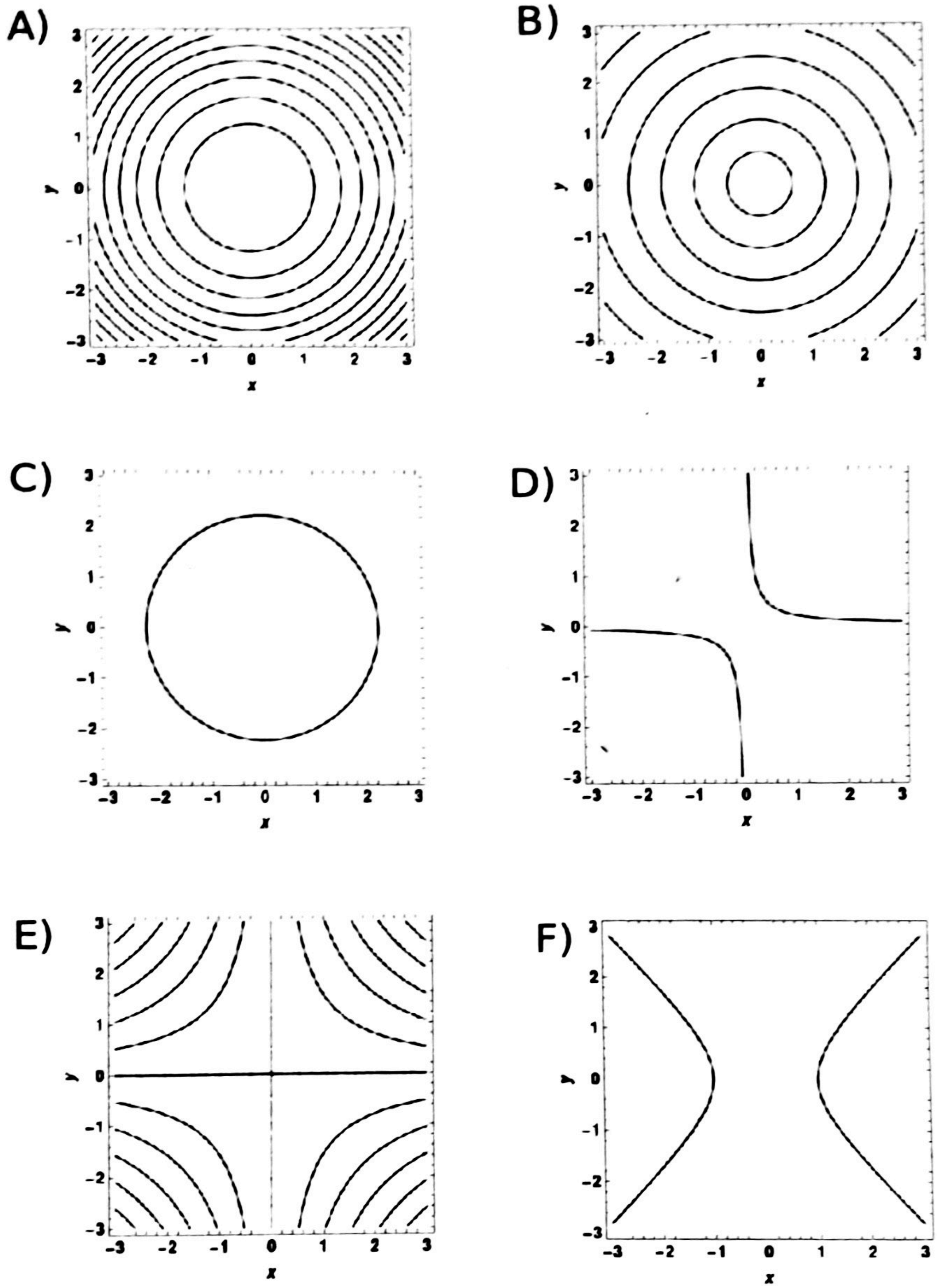

Figure 1

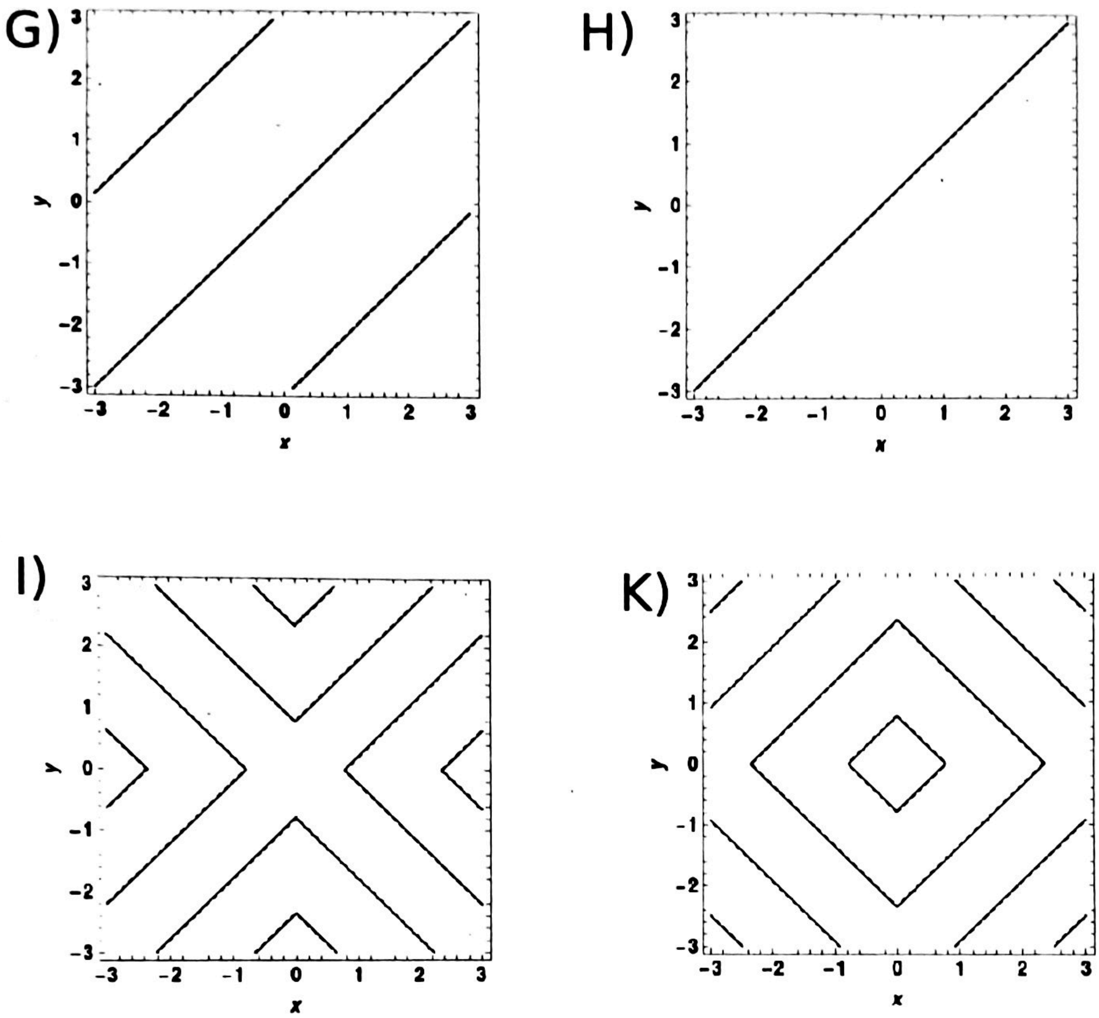

Figure 2