# Midterm 2

**Instructions:**

- This is a take-home exam, open book and open notes. You have 24 hours to complete it. You have to submit your solutions **on gradescope by 8am on Thursday November 26, 2020**.
- Your exam should be **hand-written** (electronic pencil on a tablet is allowed).
- Please start a **new page for each question**. When uploading to gradescope, you will be asked to indicate for each question on which page(s) it can be found.
- To keep answers consistent, if you have questions while working on this exam, you should direct them by email to me: `pspaas@math.ucla.edu`. In particular, your TA will just forward any emails he gets to me, hence there is no need to email them.
  *Only questions about clarification of questions will be considered, no hints will be given.*
- We refer to the course syllabus for exam policies regarding academic integrity, and want to remind you that any violations will be taken seriously. Note that you have to copy and sign the academic integrity statement contained in question 1. Failing to do so may result in your exam receiving a failing grade.
- As usual, you must **show all your work** to receive credit. Correct answers without justification will not be awarded any points.
- Good luck!!!

| Question | Points | Score |
|---|---|---|
| 1 | 7 | |
| 2 | 8 | |
| 3 | 7 | |
| 4 | 6 | |
| 5 | 6 | |
| 6 | 6 | |
| Total: | 40 | |

1. (a) Copy the following statement on your solution, and sign it with your **full name**, **UID**, and **signature**.

   *I certify on my honor that I have neither given nor received any help, and that I have not used any non-permitted resources, while completing this assignment.*

   At time $t = 0$ a skydiver jumps out of a plane. Gravity together with some heavy winds make his velocity vector at time $t$ equal to

$$\mathbf{v}(t) = \langle 2, 3t^2, -4t \rangle.$$

   (b) (3 points) Assuming the skydiver jumped out of the plane at the initial point $(0, 0, 8)$, find the parametrization $\mathbf{r}(t)$ of the curve our skydiver traverses.

   (c) (4 points) Find the unit normal vector $\mathbf{N}$ to the curve at time $t = 1$.

**Solution:**

(b) By integrating the velocity, we get

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \langle 2t, t^3, -2t^2 \rangle + \mathbf{c}$$

for some constant vector $\mathbf{c}$. Plugging in the initial condition, we get

$$\langle 0, 0, 8 \rangle = \mathbf{r}(0) = \mathbf{c}.$$

Hence $\mathbf{r}(t) = \langle 2t, t^3, 8 - 2t^2 \rangle$.

(c) Firstly, we see that

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 6t, -4 \rangle,$$

hence $\mathbf{a}(1) = \langle 0, 6, -4 \rangle$. Also, we know that $\mathbf{v}(1) = \langle 2, 3, -4 \rangle$. Since $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N} = \mathbf{a}_{\|\mathbf{v}} + \mathbf{a}_{\|\mathbf{N}}$, we get that

$$\begin{aligned} a_{\mathbf{N}}\mathbf{N}(1) &= \mathbf{a}(1) - \mathbf{a}(1)_{\|\mathbf{v}(1)} \\ &= \langle 0, 6, -4 \rangle - \frac{\langle 0, 6, -4 \rangle \bullet \langle 2, 3, -4 \rangle}{\langle 2, 3, -4 \rangle \bullet \langle 2, 3, -4 \rangle} \langle 2, 3, -4 \rangle \\ &= \frac{4}{29} \langle -17, 18, 5 \rangle. \end{aligned}$$

Since $\mathbf{a}_{\mathbf{N}} \geq 0$, we get that

$$\mathbf{N}(1) = \frac{\langle -17, 18, 5 \rangle}{\|\langle -17, 18, 5 \rangle\|} = \frac{1}{\sqrt{638}} \langle -17, 18, 5 \rangle.$$

2. For each of the following limits, calculate it, or show that it doesn't exist.

(a) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x)\sin(y)}{x^2+y^2}$.

(b) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4+y^4}$.

**Solution:**

(a) By calculating the limit along the path $x = 0$, we find that the limit equals zero. On the other hand, along the path $y = x$, we get

$$\lim_{x\to 0} \frac{\sin(x)^2}{2x^2} = \frac{1}{2},$$

since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ by applying l'Hôpital's rule. Since these two paths give different results, the limit does not exist.

(b) Since $x^4$ is always positive, we see that

$$\left|\frac{x^4y^4}{x^4+y^4}\right| \leq \left|\frac{x^4y^4}{y^4}\right| = \left|x^4\right|.$$

Hence

$$-\left|x^4\right| \leq \frac{x^4y^4}{x^4+y^4} \leq \left|x^4\right|.$$

Since

$$\lim_{(x,y)\to(0,0)} \pm\left|x^4\right| = 0,$$

it follows from the squeeze theorem that also

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4+y^4} = 0.$$

3. Consider the differentiable function $f(x, y) = y^2e^x$.

(a) (3 points) Find an equation of the tangent plane to the graph of $f$ at $(0, 2, f(0, 2))$.

(b) (2 points) Use an appropriate linear approximation to estimate the value $2.08^2e^{-0.1}$.

(c) (2 points) Find all points on the graph of $f$ where the tangent plane is parallel to the plane $x - z = 3$, or explain why no such points exist.

**Solution:**

(a) We can calculate:

$$f(0,2) = 4,$$
$$f_x(0,2) = 4,$$
$$f_y(0,2) = 4.$$

Thus, the equation for the tangent plane is given by

$$\begin{aligned} z = L(x,y) &= f(0,2) + f_x(0,2)(x-0) + f_y(0,2)(y-2) \\ &= 4 + 4x + 4(y-2) = 4x + 4y - 4. \end{aligned}$$

(b) We can use the linearization from part (a), to get that

$$f(-0.1, 2.08) \approx L(-0.1, 2.08) = 4 + 4(-0.1) + 4(0.08) = 3.92.$$

(c) If the tangent plane is parallel to $x - z = 3$, its normal vector must be parallel to $\langle 1, 0, -1\rangle$. From the equation of the tangent plane, we see that a normal vector to the tangent plane at $(a, b, f(a,b))$ is given by

$$\langle f_x(a,b), f_y(a,b), -1\rangle.$$

If this vector is a multiple of $\langle 1, 0, -1\rangle$, we necessarily have $f_x(a,b) = 1$ and $f_y(a,b) = 0$. Plugging in the equations, this gives

$$f_x(a,b) = b^2 e^a = 1 \qquad \text{and} \qquad f_y(a,b) = 2be^a = 0.$$

Since the second equation yields $b = 0$, which violates the first equation, we conclude that no such points exist.

4. Consider the function defined by $f(x,y) = (xy^2)^{1/3}$.

   (a) (2 points) Calculate $f_x(0,0)$ and $f_y(0,0)$.

   (b) (3 points) Calculate the directional derivative of $f$ at $(0,0)$ in the direction of $\mathbf{v} = \langle 1, 1\rangle$ using the limit definition of the directional derivative.

   (c) (1 point) Is $f(x,y)$ everywhere differentiable? Explain.

**Solution:**

(a) Since $f(x,0) = 0$ for all $x$, and $f(0,y) = 0$ for all $y$, we get that $f_x(0,0) = f_y(0,0) = 0$.

(b) The unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$. Therefore, by definition, the directional derivative is

$$\begin{aligned} D_{\mathbf{u}}f(0,0) &= \lim_{t\to 0} \frac{f(0+t/\sqrt{2}, 0+t/\sqrt{2}) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{t/\sqrt{2}}{t} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

(c) Since $D_{\mathbf{u}}f(0,0) \neq \nabla f(0,0) \bullet \mathbf{u}$, we conclude that the function is not differentiable. (Because if it were, this should be an equality by a theorem we saw in lecture.)

5. Consider the function $f(x, y) = x^3 + xy$.
   (a) (1 point) Calculate $\nabla f(x, y)$.
   (b) (1 point) Is $f(x, y)$ everywhere differentiable? Explain.
   (c) (1 point) Standing at the point $(1, -1)$ in the $xy$-plane, in which direction do we have to walk in order to see the value of $f(x, y)$ decrease the fastest?
   (d) (3 points) Sketch the level curve $f(x, y) = 0$ in the $xy$-plane, and draw the vector $\nabla f(1, -1)$ based at $(1, -1)$ in the same sketch.

**Solution:**

(a) $\nabla(f) = \langle f_x, f_y\rangle = \langle 3x^2 + y, x\rangle$.

(b) Yes, because its partial derivatives are everywhere continuous, which allows us to apply a theorem from lecture.

(c) We have to walk in the opposite direction of the gradient, i.e. $-\nabla f(1, -1) = -\langle 2, 1\rangle$.

(d) The level curve is given by $x^3 + xy = 0$, which we can rewrite as $x(x^2 + y) = 0$. Thus the level curve consists of the $y$-axis (where $x = 0$) together with the parabola $y = -x^2$. The vector we have to draw is the vector $\langle 2, 1\rangle$ based at the point $(1, -1)$.

6. Suppose $f(x, y)$ is a differentiable function.
   (a) (3 points) Suppose that $\nabla f(3, 5) = \langle -1, 6\rangle$, and that $\mathbf{v}$ is a vector with magnitude 3 that makes an angle of $3\pi/4$ with $\nabla f(3, 5)$. Find the directional derivative of $f$ at $(3, 5)$ in the direction of $\mathbf{v}$.

(b) (3 points) Suppose that at the point $(2, 1)$, the tangent line to the level curve of $f(x, y)$ that goes through $(2, 1)$ has equation $-2x + 3y + 1 = 0$. Furthermore, assume that $f_x(2, 1) > 0$. Find the unit vector that points in the same direction as $\nabla f(2, 1)$.

**Solution:**

(a) Write $\mathbf{u}$ for the unit vector in the direction of $\mathbf{v}$. Then since $f$ is differentiable, the directional derivative of $f$ in the direction of $\mathbf{v}$ is given by

$$\nabla f(3, 5) \bullet \mathbf{u} = \|\nabla f(3, 5)\| \cos(\theta) = \sqrt{37}(-\sqrt{2}/2) = -\frac{\sqrt{74}}{2}.$$

(b) Since we know that the gradient has to be perpendicular to the level curve, it has to be parallel to the vector $\langle -2, 3 \rangle$ (as this vector is perpendicular to the direction vector of the given tangent line). Since we also know that $f_x(2, 1) > 0$, i.e. the $x$-component of $\nabla f(2, 1)$ is positive, $\nabla f(2, 1)$ has to point in the direction of $\langle 2, -3 \rangle$. The unit vector in this direction is

$$\frac{\langle 2, -3 \rangle}{\|\langle 2, -3 \rangle\|} = \frac{1}{\sqrt{13}}\langle 2, -3 \rangle.$$