Midterm 1

Instructions: (READ CAREFULLY—YOU HAVE THE TIME FOR IT.)

- This is a take-home exam, open book and open notes. You have 24 hours to complete it. You have to submit your solutions **on gradescope by 8am on Thursday October 29, 2020**.
- Your exam should be **hand-written** (electronic pencil on a tablet is allowed).
- Please start a **new page for each question**. When uploading to gradescope, you will be asked to indicate for each question on which page(s) it can be found.
- To keep answers consistent, if you have questions while working on this exam, you should direct them by email to me: `pspaas@math.ucla.edu`. In particular, your TA will just forward any emails he gets to me, hence there is no need to email them.
  Only questions about clarification of questions will be considered, no hints will be given.
- We refer to the course syllabus for exam policies regarding academic integrity, and want to remind you that any violations will be taken seriously. Note that you have to copy and sign the academic integrity statement contained in question 1. Failing to do so may result in your exam receiving a failing grade.
- As usual, you must **show all your work** to receive credit. Correct answers without justification will not be awarded any points.
- Good luck!!!

Question	Points	Score
1	8	
2	6	
3	7	
4	6	
5	7	
6	6	
Total:	40	

1. (a) Copy the following statement on your solution, and sign it with your **full name**, **UID**, and **signature**.

   I certify on my honor that I have neither given nor received any help, and that I have not used any non-permitted resources, while completing this assignment.

   Consider the triangle formed by the points $P = (1, 0, 0)$, $Q = (1, 2, 6)$, and $R = (1, -3, 1)$.

   (b) (2 points) Show that this triangle has a right angle at P.

   (c) (3 points) Find the area of this triangle.

   (d) (3 points) Find a parametrization of the line through P and Q.

Solution:

(b) For this we need to check that the angle between $\mathbf{v} = \overrightarrow{PQ}$ and $\mathbf{w} = \overrightarrow{PR}$ is $\pi/2$, hence we need to calculate the dot product. We see that

$$\mathbf{v} \bullet \mathbf{w} = \langle 0, 2, 6 \rangle \bullet \langle 0, -3, 1 \rangle = 0.$$

Hence these vectors are indeed perpendicular.

(c) The area is given by $\frac{\|\mathbf{v}\times\mathbf{w}\|}{2}$. We see that

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 6 \\ 0 & -3 & 1 \end{vmatrix} = \langle 20, 0, 0 \rangle.$$

Hence the area is $\frac{\|\mathbf{v}\times\mathbf{w}\|}{2} = 10$.

(d) The direction vector for this line is $\mathbf{v} = \langle 0, 2, 6 \rangle$ and it goes (for instance) through the point P, so a parametrization of this line is given by

$$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t\langle 0, 2, 6 \rangle.$$

2. (a) (2 points) Are the points $P = (1, 2, 0)$, $Q = (-1, 2, 3)$ and $R = (0, -2, 5)$ collinear? Explain.

   (b) (4 points) Find the equation of the plane that contains the lines given by $\mathbf{r}_1(t) = \langle 0, 1, 2 \rangle + t\langle -1, 2, 0 \rangle$ and $\mathbf{r}_2(t) = \langle -1, 3, 2 \rangle + t\langle 0, 2, -4 \rangle$.

Solution:

(a) We see that $\overrightarrow{PQ} = \langle -2, 0, 3 \rangle$ and $\overrightarrow{PR} = \langle -1, -4, 5 \rangle$. Since these vectors are not parallel (they are not scalar multiples of each other), we conclude that the points are not collinear. (You can also do this in different ways, for instance checking if R lies on the line through P and Q.)

(b) From the assumptions, we get that the vectors $\langle -1, 2, 0\rangle$ and $\langle 0, 2, -4\rangle$ are parallel to the plane, hence their cross product will be a normal vector. We calculate

$$\langle -1, 2, 0\rangle \times \langle 0, 2, -4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & 2 & -4 \end{vmatrix} = \langle -8, -4, -2\rangle.$$

Hence an equation of the plane is $-8x - 4y - 2z = -8$, where we found the value -8 by plugging in one of the points in the plane, for instance $(0, 1, 2)$.

3. Suppose $\mathbf{r}_1(t) = \langle t^2, 3e^t, \sin(\pi t)\rangle$.

   (a) (3 points) Find a parametrization of the tangent line to this curve at $t = 3$.

   (b) (4 points) Suppose $\mathbf{r}_2(t)$ is another curve which satisfies $\mathbf{r}_2(3) = \langle 1, 3, 0\rangle$ and $\mathbf{r}_2'(t) = \langle 4t, 0, t^2\rangle$. Find $\frac{d}{dt}(\mathbf{r}_1(t) \bullet \mathbf{r}_2(t))$ at $t = 3$.

Solution:

(a) We know the derivative gives a tangent vector, hence a direction vector for the tangent line. We calculate:

$$\mathbf{r}'(t) = \langle 2t, 3e^t, \pi\cos(\pi t)\rangle.$$

Hence a parametrization for the tangent line at $t = 3$ is given by

$$\mathbf{L}(t) = \mathbf{r}(3) + t\,\mathbf{r}'(3) = \langle 9, 3e^3, 0\rangle + t\,\langle 6, 3e^3, -\pi\rangle.$$

(b) From the product rule, we get that

$$\frac{d}{dt}(\mathbf{r}_1(t) \bullet \mathbf{r}_2(t)) = \mathbf{r}_1'(t) \bullet \mathbf{r}_2(t) + \mathbf{r}_1(t) \bullet \mathbf{r}_2'(t).$$

At $t = 3$, we thus find

$$\begin{aligned} &\mathbf{r}_1'(3) \bullet \mathbf{r}_2(3) + \mathbf{r}_1(3) \bullet \mathbf{r}_2'(3) = \\ &\qquad = \langle 6, 3e^3, -\pi\rangle \bullet \langle 1, 3, 0\rangle + \langle 9, 3e^3, 0\rangle \bullet \langle 12, 0, 9\rangle \\ &\qquad = 6 + 9e^3 + 108 = 114 + 9e^3. \end{aligned}$$

4. Consider the curve $\mathcal{C}$ parametrized by $\mathbf{r}(t) = \langle t^3, 5, 2t^3\rangle$ for $1 \le t \le 5$.

(a) (3 points) Is $\mathbf{r}(t)$ the arc length parametrization of $\mathcal{C}$? Explain why/why not.

(b) (3 points) Find the arc length of the curve for $1 \leq t \leq 5$.

Solution:

(a) If it is the arc length parametrization, we should have $\|\mathbf{r}'(t)\| = 1$ for all t. We calculate:

$$\mathbf{r}'(t) = \langle 3t^2, 0, 6t^2 \rangle,$$

and hence

$$\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 36t^4} = \sqrt{45}t^2.$$

Since this is not always equal to 1 (for say $t = 1$ it is equal to $\sqrt{45}$), this is not the arc length parametrization.

(b) Using the calculations from part (a), we get that the arc length is equal to

$$\begin{aligned}
\int_1^5 \|\mathbf{r}'(t)\| \, dt &= \int_1^5 \sqrt{45}t^2 \, dt \\
&= \sqrt{45} \left(\frac{t^3}{3} \right)_1^5 \\
&= \sqrt{45} \, \frac{124}{3} \\
&= 124\sqrt{5}.
\end{aligned}$$

5. Consider the curve parametrized by $\mathbf{r}(t) = \langle t\cos(t) - \sin(t), t\sin(t) + \cos(t), t^2 \rangle$.

(a) (5 points) Find expressions for the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$.

(b) (2 points) Find the curvature of $\mathbf{r}(t)$ at $t = 2$.

Solution:

(a) We calculate:

$$\mathbf{r}'(t) = \langle \cos(t) - t\sin(t) - \cos(t), \sin(t) + t\cos(t) - \sin(t), 2t \rangle = \langle -t\sin(t), t\cos(t), 2t \rangle,$$

and

$$\|\mathbf{r}'(t)\| = \sqrt{t^2\sin^2(t) + t^2\cos^2(t) + 4t^2} = \sqrt{5}t.$$

Hence

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{5}} \langle -\sin(t), \cos(t), 2 \rangle.$$

Similarly, we get

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{1}{\sqrt{5}}\langle -\cos(t), -\sin(t), 0\rangle}{\frac{1}{\sqrt{5}}} = \langle -\cos(t), -\sin(t), 0\rangle.$$

(b) We can use the formula $\kappa(t) = \dfrac{1}{v(t)}\,\|\mathbf{T}'(t)\|$, where $v(t) = \|\mathbf{r}'(t)\|$. From part (a), this gives

$$\kappa(t) = \frac{1}{\sqrt{5}t}\,\frac{1}{\sqrt{5}}.$$

Thus, for $t = 2$, this gives $\kappa(2) = \dfrac{1}{10}$.

6. (a) (3 points) Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors based at the origin in $\mathbb{R}^3$ such that $\mathbf{v}$ and $\mathbf{w}$ are contained in the xy-plane, $\|\mathbf{v} \times \mathbf{w}\| = 3$, and $\mathbf{u}$ is a vector of length 2 contained in the first quadrant of the yz-plane that makes an angle of $\dfrac{\pi}{6}$ with the y-axis. Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

(b) (3 points) Do there exist *nonzero* vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ such that $\mathbf{v} \bullet \mathbf{w} = 0$ and $\mathbf{v} \times \mathbf{w} = \mathbf{0}$? If so, give an example. If not, explain why not.

Solution:

(a) We know that the volume of the required parallelepiped is equal to $|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| = \|\mathbf{u}\|\,\|\mathbf{v} \times \mathbf{w}\|\,|\cos(\theta)|$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$. Since $\mathbf{v}$ and $\mathbf{w}$ are both contained in the xy-plane, their cross product is parallel to the z-axis. Thus, we need to find the angle between $\mathbf{u}$ and the z-axis. Since $\mathbf{u}$ lies in the first quadrant of the yz-plane and makes an angle of $\pi/6$ with the y-axis, it makes an angle of $\pi/3$ with the positive z-axis. Hence the angle θ is either equal to $\pi/3$ (if $\mathbf{v} \times \mathbf{w}$ points in the positive z-direction) or equal to $2\pi/3$ (if $\mathbf{v} \times \mathbf{w}$ points in the negative z-direction). In either case, $|\cos(\theta)| = 1/2$ and we find

$$|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| = \|\mathbf{u}\|\,\|\mathbf{v} \times \mathbf{w}\|\,|\cos(\theta)| = 2 \cdot 3 \cdot \frac{1}{2} = 3.$$

(b) No. If $\mathbf{v} \bullet \mathbf{w} = 0$, then $\mathbf{v}$ and $\mathbf{w}$ have to be perpendicular, and if $\mathbf{v} \times \mathbf{w} = \mathbf{0}$, then $\mathbf{v}$ and $\mathbf{w}$ have to be parallel. These cannot both happen at the same time.