Midterm 1

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UID: [redacted]

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Yurun Ge
	1C	1D	TA: Benjamin Johnsrude
	1E	1F	TA: Tianqi Wu

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. You may not use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. **Do not write answers for one question on the page of another question.** If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

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1. Given $\mathbf{u} = \langle 1, 2, 0\rangle, \mathbf{v} = \langle -3, -1, 0\rangle$, and $\mathbf{w} = \langle -2, 1, 5\rangle$.
   (a) (3 points) Is $\mathbf{w}$ perpendicular to $\mathbf{u}$ and/or $\mathbf{v}$? Explain.
   (b) (4 points) Calculate the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.
   (c) (3 points) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{w}$.

(a) $\vec{w} \cdot \vec{u} = -2 + 2 + 0 = 0$

$\vec{w} \cdot \vec{v} = 6 - 1 + 0 = 5$

↳ Vectors are perpendicular only when their dot product is 0.
$\vec{w}$ is perpendicular to $\vec{u}$, but not $\vec{v}$.

(b) $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

$\Leftrightarrow -3 - 2 = (\sqrt{5})(\sqrt{10})\cos\theta$

$\Leftrightarrow \cos\theta = \dfrac{-5}{\sqrt{50}} = -\dfrac{5}{5\sqrt{2}} = -\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2} \Leftrightarrow \theta = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

$\Leftrightarrow \boxed{\theta = \dfrac{3\pi}{4}}$ (or the obtuse angle $= \dfrac{5\pi}{4}$)

(c) Area $= \|\vec{u} \times \vec{w}\|$

$\vec{u} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ -2 & 1 & 5 \end{vmatrix} = \langle 10, -5, 5\rangle$

↳ $\|\vec{u} \times \vec{w}\| = \|\langle 10, -5, 5\rangle\| = \sqrt{100 + 25 + 25}$

$= \sqrt{150} = \boxed{5\sqrt{6} \text{ units}^2}$

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2. Let $P = (1,0,0), Q = (1,1,1), R = (0,1,3), S = (1,5,3)$.

(a) (6 points) Show that P, Q, R are not collinear and find an equation of the plane containing P, Q, R.

(b) (1 point) Show that there is no plane in $\mathbb{R}^3$ that contains all four of P, Q, R, S.

(c) (3 points) Find a parametrization of the intersection between the plane from part (a) and the surface $4x^2 + 2z^2 = 1$.

(a) Not collinear: P, Q have same x-coordinate $(x=1)$, R doesn't $(x=0)$.

When 2 of 3 pts but not all 3 share the same x-coord, the 3 pts are not collinear.

↳ Also, $\vec{PQ}$ and $\vec{PR}$ are not multiples of each other! (vectors found below) ($\vec{PQ} \times \vec{PR} \neq \vec{0}$)

Eqn of plane: $\vec{PQ} = \langle 0, 1, 1 \rangle$

$\vec{PR} = \langle -1, 1, 3 \rangle$

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{vmatrix} = \langle 2, -1, 1 \rangle$ (normal vector)

↳ Eqn: $2x - y + z = d \rightarrow 2 - 0 + 0 = d \rightarrow d = 2$

↳ $\boxed{2x - y + z = 2}$

(b) There's only one plane containing P, Q, R and its given by the above eqn (P, Q, R aren't collinear). Thus, for S to also be coplanar, it must lie on the plane found in (a). However: $2(1) - 5 + 3 = 0 \neq 2$. Therefore, there's no plane in $\mathbb{R}^3$ containing P, Q, R, and S.

(c) $x: \frac{1}{2}\cos(t)$

$z: \frac{1}{\sqrt{2}}\sin(t)$

parametrization of $4x^2 + 2z^2 = 1$

$y = 2x + z - 2 = \cos(t) + \frac{1}{\sqrt{2}}\sin(t) - 2$

↳ Parametrization: $\begin{cases} x: \frac{1}{2}\cos(t) \\ y: \cos(t) + \frac{1}{\sqrt{2}}\sin(t) - 2 \\ z: \frac{1}{\sqrt{2}}\sin(t) \end{cases}$

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3. Suppose $\mathbf{r}_1(t) = \langle e^t, \cos(\pi t), 2t\rangle$, and $\mathbf{r}_2(t) = \langle 3t^2, t^2, 0\rangle$.

(a) (5 points) Find the unit tangent vector and a parametrization of the tangent line to the curve parametrized by $\mathbf{r}_1$ at $t = 1$.

(b) (5 points) Find the arc length of the curve parametrized by $\mathbf{r}_2$ for $0 \le t \le 3$.

(a) $r_1'(t) = \langle e^t, -\pi \sin(\pi t), 2\rangle$ $\quad T(t) = \dfrac{\langle e^t, -\pi \sin(\pi t), 2\rangle}{\sqrt{e^{2t} + \pi^2 \sin^2(\pi t) + 4}}$

$r_1'(1) = \langle e, 0, 2\rangle$

$$T(1) = \frac{\langle e, 0, 2\rangle}{\sqrt{e^2+4}} = \boxed{\left\langle \frac{e}{\sqrt{e^2+4}}, 0, \frac{2}{\sqrt{e^2+4}} \right\rangle}$$

Tangent line to curve @ $t=1$:

$r_1(1) = \langle e, -1, 2\rangle$

$r_1'(1) = \langle e, 0, 2\rangle$

↳ Tangent line eqn $\vec{\ell}(s) = \langle e + e(s), -1, 2 + 2(s)\rangle$

(or more simply, $\boxed{\vec{\ell}(u) = \langle e(u), -1, 2u\rangle}$ for $u = s+1$)

(b) Arc length $0 \le t \le 3$ $= \displaystyle\int_0^3 \| r_2'(t) \| \, dt$

$r_2'(t) = \langle 6t, 2t, 0\rangle$

$\| r_2'(t) \| = \sqrt{36t^2 + 4t^2} = \sqrt{40t^2} = 2\sqrt{10}\, t$

↳ Arc length $= \displaystyle\int_0^3 2\sqrt{10}\, t \, dt = \sqrt{10}\, t^2 \Big|_0^3 = \sqrt{10}(9) - 0$

$= \boxed{9\sqrt{10}}$

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4. At $t = 0$ a skydiver jumps out of a plane. Gravity together with some heavy winds make his velocity vector at time t equal to

$$\mathbf{v}(t) = \langle 4 \sin 2t, 4 \cos 2t, -4t \rangle.$$

(a) (5 points) Assuming he jumped out of the plane at the initial point $(0, 0, 8)$, find the parametrization $\mathbf{r}(t)$ of the curve our skydiver traverses.

(b) (2 points) At what speed does the skydiver hit the xy-plane?

(c) (3 points) Is the parametrization $\mathbf{r}$ you found in part (a) the arc length parametrization? Explain why or why not.

(a) $\vec{r}(t) = \int \vec{v}(t)\, dt = \int \langle 4 \sin 2t, 4 \cos 2t, -4t \rangle\, dt = \langle -2\cos 2t, 2\sin 2t, -2t^2 \rangle + \vec{c}$

$\vec{r}(0) = \langle 0, 0, 8 \rangle = \langle -2, 0, 0 \rangle + \vec{c} \iff \vec{c} = \langle 2, 0, 8 \rangle$

$\hookrightarrow \boxed{\vec{r}(t) = \langle 2 - 2\cos(2t), 2\sin(2t), 8 - 2t^2 \rangle}$

(b) Hits xy plane when $z = 0$:

$z(t) = 8 - 2t^2 = 0 \iff 2t^2 = 8 \iff t^2 = 4 \iff t = \pm 2$

But $t \geq 0$, so $t = 2$

Velocity: $\vec{v}(2) = \langle 4 \sin(4), 4\cos(4), -8 \rangle$

Speed: $v(2) = \| \langle 4 \sin(4), 4\cos(4), -8 \rangle \| = \sqrt{16 + 64} = \sqrt{80} = \boxed{4\sqrt{5}}$

(c) No it is not the arc length parametrization. The arc length parametrization traverses the curve at speed 1 whereas the $\vec{r}(t)$ I found doesn't — rather, it traverses at speed $4\sqrt{1+t^2}$. For instance, as we found in (b), when $t=2$, speed $v = 4\sqrt{5} \neq 1$.