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Math 32A - Final

1. a. I certify on my honor that I have neither given nor received help and that I have not used any non-permitted resources while completing this assignment.

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b. $\vec{u} = \langle 1, -2, 0 \rangle$, $\vec{v} = \langle 4, -3, 6 \rangle$

$$\vec{v}_{\parallel \vec{u}} = \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\right)\vec{u} = \frac{10}{61}\vec{u} = \left\langle \frac{40}{61}, \frac{-30}{61}, \frac{60}{61} \right\rangle$$

$$\vec{v}_{\perp \vec{u}} = \vec{u} - \vec{v}_{\parallel \vec{u}} = \left\langle \frac{21}{61}, \frac{\cancel{-152}\ -92}{61}, \frac{-60}{61} \right\rangle$$

c. The greatest volume of a parallelepiped occurs when it is a cuboid. This ~~is~~ results in ~~x~~ $\vec{u}\times\vec{w}$ having the greatest surface area ($\sin\frac{\pi}{2}$ is the max sine value) and being multiplied by $\|\vec{v}\|$ directly ~~ratht~~ rather than $\|\vec{v}_{\parallel(\vec{u}\times\vec{w})}\|$ which is $\leq \|\vec{v}\|$. Thus:

$$\|\vec{u}\times\vec{w}\| = \|\vec{u}\|\|\vec{w}\| \sin\frac{\pi}{2} = 6$$

$$\|\vec{v}\| = \sqrt{2^2+1^2+2^2} = 3$$

$\therefore$ max volume $= \|\vec{v}\|\|\vec{u}\times\vec{w}\| = 18$

$\therefore$ the volume is always ≤ 18.

2 a $v(t) = \|r'(t)\|$ $\qquad \vec{r}\,'(t) = \langle 12t - 1, 13, 3t^2 - 4t + 3 \rangle$

$\vec{r}\,'(2) = \langle 23, 13, 7 \rangle$

$\therefore v(t) = \sqrt{23^2 + 13^2 + 7^2} = 3\sqrt{83}$

b. $\vec{r}(2) = \langle 22, 26, 6 \rangle$

Tangent line:

$l_1 = \langle 22, 26, 6 \rangle + \lambda \langle 23, 13, 7 \rangle, \quad \lambda \in \mathbb{R}$

OR

$l_1(\lambda) = \langle 23\lambda + 22, 13\lambda + 26, 7\lambda + 6 \rangle$

$r'(t) =$

c. $\langle 12t-1, 13, 3t^2-4t+3 \rangle = \langle -2, 26, 0 \rangle = \langle -1, 13, 0 \rangle$

(13 = 13)

$12t - 1 = -2$ and $3t^2 - 4t + 3 = 0$

$t = \frac{-1}{12}$ $\sqrt{b^2 - 4ac} \rightarrow \sqrt{16 - 4 \times 3 \times 3}$

$= \sqrt{-20}$

↳ no real solutions

$\therefore$ There is no time t where the plane is flying in the direction $\langle -2, 26, 0 \rangle$

3a. $\vec{r}'(t) = \langle 2t-1, \cos t \rangle$

$\|r'(t)\| = \sqrt{4t^2 - 4t + 1 + \cos^2 t}$

It can be observed that $\|r'(t)\| \neq 1$ for all t, so it is not the arc length parametrization.

Eg $\rightarrow$ $t = 0$ $\longrightarrow$ $\|r'(t)\| = \sqrt{2}$

b. $\frac{d}{dt}(f(\vec{r}(t)))\Big|_{t=0} = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$

$r(0) = \langle 0, 1\cancel{8} \rangle$, $\nabla f(0,1) = \langle 3, -2 \rangle$

$r'(0) = \langle -1, 1 \rangle$

$\frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} = \langle 3, -2 \rangle \cdot \langle -1, 1 \rangle = -5$

c. $\frac{\partial f}{\partial v} = \cancel{\frac{\partial f}{\partial x}} \nabla f \cdot \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle, \Big|_{u,v = 1,0}$

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$\frac{\partial x}{\partial v} = 2v$, $\frac{\partial y}{\partial v} = u$

At $(u, v) = (1, 0)$ $x = v^2 = 0$, $y = u^2 + uv = 1$

$\frac{\partial f}{\partial v} = \nabla f_{(0,1)} \cdot \langle 2v, u \rangle$

$= \langle 3, -2 \rangle \cdot \langle 0, 1 \rangle$

$= -2$

4. $r_1(t) = \langle 1+2t, -3t, 2-t \rangle$

~~$r_2(s) = \langle 3, -5, 5,$~~

$r_2(n) = \langle 3, -5+n, 7-3n \rangle$

$1+2t = 3 \longrightarrow t = 1$

$-3t = -5+n \longrightarrow -3 = -5+n \longrightarrow n = 2$

$7 - 3 \cdot 2 = 2 - 1$

$1 = 1 \longrightarrow$ there is an intersection at $t=1$ and $n=2$

$t=1$

$r_1(1) = \langle 3, -3, 1 \rangle$

$r_1(t) \cdot r_2(n) \Rightarrow \vec{r}_1$ direction vector $= \langle 2, -3, -1 \rangle$

$\vec{r}_2$ direction vector $= \langle 0, 1, -3 \rangle$

$\|r_1\| = \sqrt{2^2+3^2+1} = \sqrt{14}$

$\|r_2\| = \sqrt{3^2+1} = \sqrt{10}$

$r_1 \cdot r_2 = 2 \cdot 0 + 1 \cdot -3 - 1 \cdot -3$

$= 0$

$\cos\theta = \dfrac{0}{\sqrt{10}\sqrt{14}} = 0$

$\theta = \dfrac{\pi}{2}$

6. $r_1 \times r_2 \rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & -1 \\ 0 & 1 & -3 \end{vmatrix} = \langle 10, 6, 2 \rangle = \vec{n}$

$\langle 10, 6, 2 \rangle \langle x-3, y+3, z-1 \rangle = 0$

$10x - 30 + 6y + 18 + 2z - 2 = 0$

$10x + 6y + 2z = 14$

$5x + 3y + z = 7$

c. ~~$L(x,y) = 10(x-3) +$~~

$10(x-3) + 6(y+3) + 2(z-1) = 0$

~~$z = 1 - 10(x-3) - 6(y+3)$~~

$z = 1 - 5(x-3) - 3(y+3) = L(x, y)$

$L(1.1, 1.9) = 1 - 9.5 - 14.7$

$= -23.2$

c. $z = 7 - 5x - 3y = L(x, y)$

$L(1.1, 1.9) = 7 - 5 \cdot 1.1 - 3 \cdot 1.9$

$= -4.2$

5a $r'(t) = \langle 8, -6, \cancel{8}\ -10t \rangle$

$r''(t) = a(t) = \langle 0, 0, -10 \rangle$

6. $a(t) = a_T T(t) + a_N N(t)$

① $\therefore a_T \cdot \frac{4}{5\sqrt{t^2+1}} + a_N \cdot \frac{-4t}{5\sqrt{t^2+1}} = 0$

$4a_T - 4t a_N = 0$

$a_T - t a_N = 0 \longrightarrow a_T = t a_N$

② $\cancel{\frac{-3}{5\sqrt{t^2+1}} a_T + \frac{3t}{}}$

$\frac{\cancel{8}\ 5t}{5\sqrt{t^2+1}} a_T + \frac{5}{5\sqrt{t^2+1}} a_N = -10$

$\frac{5t^2+5}{5\sqrt{t^2+1}} \cdot \cancel{3} a_N = -10$

$\frac{t^2+1}{\sqrt{t^2+1}} \cancel{\ } a_N = \frac{\cancel{8}}{-10} \longrightarrow a_N = \frac{\cancel{-5}\ -10}{\sqrt{t^2+1}}$

$a_T = t a_N = \cancel{\frac{-5t\ -10}{\sqrt{t^2+1}}} \quad \frac{-10t}{\sqrt{t^2+1}}$

6a $x^4 - y^3 + z^2 - 2z - 8 = 0 = F(x, y, z)$

$F_z = 2z - 2$

$F_y = -3y^2$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-3y^2}{2z-2}$$

at $(2, 2, 0)$

$$= \frac{-3 \cdot 2^2}{2 \cdot 0 - 2} = \frac{-12}{-2} = 6$$

6. Normal to xy plane $= \langle 0, 0, 1 \rangle \cdot \lambda,\ \lambda \in \mathbb{R}$

$\nabla F = \langle 4x^3, -3y^2, 2z - 2 \rangle$

$4x^3 = 0 \longrightarrow x = 0$

$-3y^2 = 0 \longrightarrow y = 0$

$2z - 2 = \lambda$ for $\lambda \in \mathbb{R}$

$\Rightarrow z \in \mathbb{R}$ subject to equation constraint

$\hookrightarrow x, y = 0 \longrightarrow z^2 - 2z - 8 = 0$

$(z + 2)(z - 4) = 0 \longrightarrow z = -2, 4$

Points are : $(0, 0, -2)$ and $(0, 0, 4)$

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7.a Consider $x = 0$

$$\lim_{(x,y)\to(0,0)} \frac{4 \cdot 0 \cdot y^4}{0^4 + y^8} = 0$$

Consider $x = y^2$

$$\lim_{(x,y)\to(0,0)} \frac{4y^4 \cdot y^4}{y^8 + y^8} = \lim_{(x,y)\to(0,0)} \frac{4y^8}{2y^8} = 2$$

Inconsistent limits $\therefore$ the limit does not exist.

b. $$\frac{y^2 - 4y + x + 3}{(x-1)^2 + (y-2)^2} = \frac{y^2 - 4y + 4 + x - 1}{(y-2)^2 + (x-1)^2} = \frac{(y-2)^2 + (x-1)}{(y-2)^2 + (x-1)^2}$$

Consider $y = 2$

$$\lim_{x\to1} \frac{(x-1)}{(x-1)^2} = \cancel{\lim_{x}} \lim_{x\to1} \frac{1}{x-1} \rightarrow \text{the limit D.N.E.}$$

$$\therefore \lim_{(x,y)\to(1,2)} \frac{y^2 - 4y + x + 3}{(x-1)^2 + (y-2)^2} \text{ does not exist.}$$

8 a. $f_x = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{3h^3}{h^3} = 3$

$f_y = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k}$

$= \lim_{k \to 0} \dfrac{-2k^3}{3k^3} = -\dfrac{2}{3}$

6. $\cancel{f_x = \dfrac{9x^2(x^2+3y^2) - 2x(3x^3 - 2y^3)}{(x^2+3y^2)^2}}$

$\cancel{f_y = \dfrac{-6y^2(x^2+3y^2) + 6y(3x^3-2y^3)}{(x^2+3y^2)^2}}$

~~f_x and f_y are continuous ~~for~~ at all points on a disk around $(0,0)$ as they are polynomial and $(x^2+3y^2)^2 \neq 0$. Thus, $f(x,y)$ is differentiable at $(0,0)$~~

$\nabla f(0,0) = \langle 3, -\frac{2}{3} \rangle$

$D_{\vec{u}} f(0,0) = \nabla f \cdot \vec{u}$

$= \langle 3, -\frac{2}{3} \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= \dfrac{3}{\sqrt{2}} - \dfrac{2}{3\sqrt{2}}$

$= \dfrac{7\sqrt{2}}{6} \neq \dfrac{1}{\sqrt{2}5}$

$f(x,y)$ is not differentiable at $(0,0)$ since $D_{\vec{u}} f(0,0) \neq \nabla f \cdot \vec{u}$ at $(0,0)$ which holds true at a differentiable point.

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a. $\nabla f = \langle -2\sin x, -1 \rangle$

$\nabla f_{(\pi/3,\, 1)} = \langle -\sqrt{3}, -1 \rangle$ (direction of largest value)

Magnitude $= \sqrt{3+1} = 2$ (value)

b. $2\cos x - y = 0$

$\nabla f(\pi/3, 1)$ is ~~at~~ ~~normal~~ perpendicular to the tangent at $(\pi/3, 1)$

$\therefore \langle -\sqrt{3}, -1 \rangle \cdot \vec{u} = 0$

$-\sqrt{3} \cdot u_x - u_y = 0$

$u_x = 1, \quad u_y = \sqrt{3}$

$\vec{u} = \langle 1, \sqrt{3} \rangle$

Let $\vec{u} \perp \nabla f$ be tangent.

$\langle -\sqrt{3}, -1 \rangle \cdot \langle u_x, u_y \rangle =$

$-\sqrt{3}\, u_x - u_y = 0$

$\vec{u} = \langle 1, -\sqrt{3} \rangle$

(alternate proof)

$y = 2\cos x$

$\frac{dy}{dx} = -2\sin x$ at $x = \pi/3 \longrightarrow -\sqrt{3}$

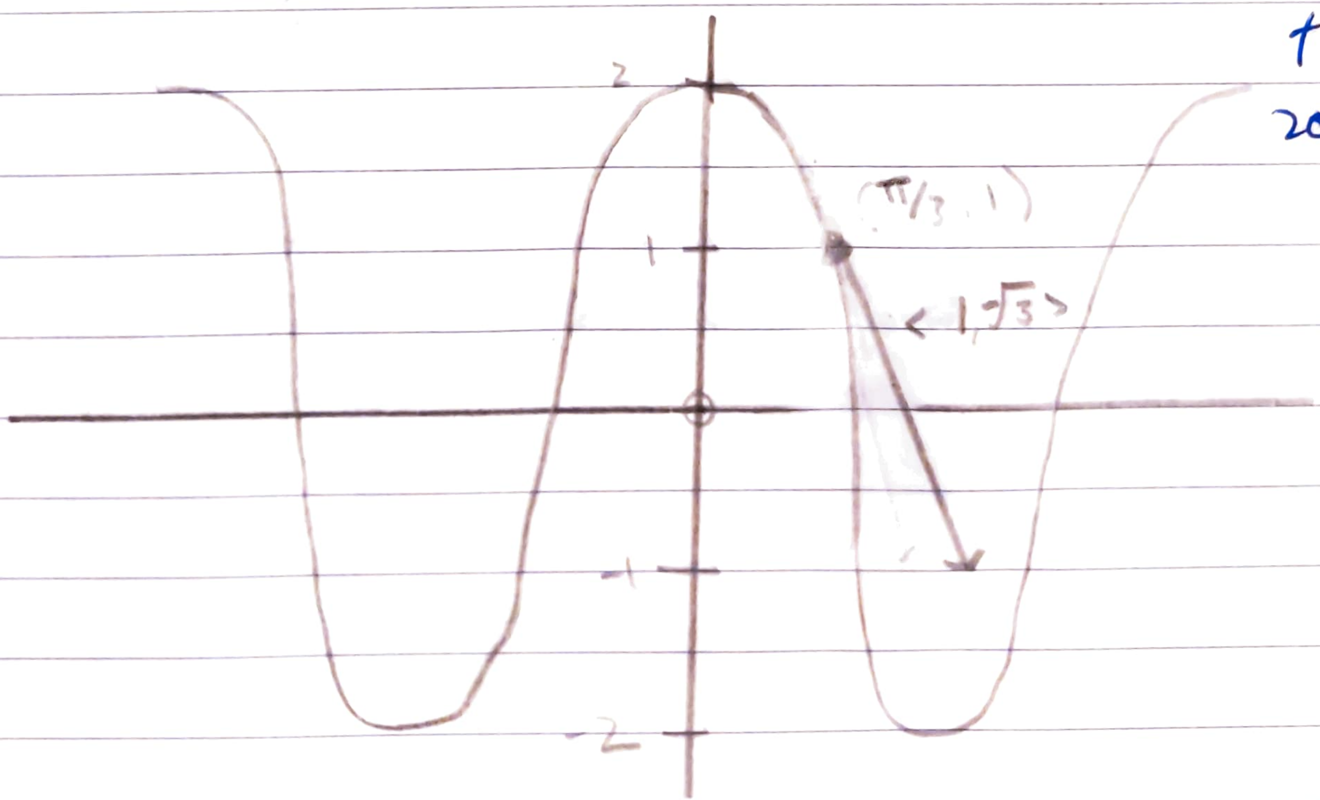
$y = -\sqrt{3}x \longrightarrow$ vector $= \langle 1, -\sqrt{3} \rangle$

$(x = 1)$

c.

$f(x, y) = 0$

$2\cos x - y = 0$



10 a. $f_x = 2xy = 0 \longrightarrow x = 0$ OR $y = 0$

$f_y = 2y - 4 + x^2 = 0$

$\cancel{2}y = \cancel{4}2 - \frac{x^2}{2}$

$\to 2x\left(2 - \frac{x^2}{2}\right) = 0$

$2 - \frac{x^2}{2} = 0 \longrightarrow x^2 = 4$, $x = \pm 2$ when $y = 0$

$x = 0 \longrightarrow y = 2$

Critical points at $(0,2), (2,0), (-2,0)$

$f_{xx} = 2y$, $f_{yy} = 2$, $f_{xy} = 2x = f_{yx}$

$$\begin{bmatrix} 2y & 2x \\ 2x & 2 \end{bmatrix} = 4y - 4x^2 = D$$

$D(0,2) = 8 > 0$ and $f_{xx}(0,2) = 4 > 0 \longrightarrow$ local ~~maxima~~ minima $(0,2)$

$D(2,0) = -16 < 0 \longrightarrow$ saddle point $(2,0)$

$D(-2,0) = -16 < 0 \longrightarrow$ saddle point $(-2,0)$

b. $x = 0 \longrightarrow f(x,y) = y^2 - 4y$

$-4y$ increases linearly, y^2 increases polynomially. Thus, since the function is unbounded and is only increasing at higher y values, it does not have a global maximum.

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11 a Boundary $\Rightarrow x^4 + y^2 = 32$.

$f(x,y)$ is polynomial and therefore continuous. ~~R~~ It is closed and bounded and therefore must have a global maximum and minimum either on critical points within D ~~or~~ or on the boundary of D.

b. $\left.\begin{array}{l} f_x = 2x \\ f_y = 1 \end{array}\right\} \nabla f \neq \vec{0}$ in D

Consider boundary using Lagrange multipliers.

$f(x,y) = x^2 + y$, $g(x,y) = x^4 + y^2 - 32 = 0$

$\nabla f = \langle 2x, 1 \rangle$

$\nabla g = \langle 4x^3, 2y \rangle = \vec{0}$ at $x=0, y=0$.

↳ does not satisfy constraint

~~OR~~ So: $\lambda \neq 0$.

$\nabla f = \lambda \nabla g$

$2x = \lambda 4x^3 \longrightarrow \lambda = \frac{1}{2x^2}$ (since $x \neq 0$) OR

$1 = \lambda 2y \longrightarrow \lambda = \frac{1}{2y}$ $(y \neq 0)$

if $x = 0$,
$y = \sqrt{32}$
$= \pm 4\sqrt{2}$

$\frac{1}{2x^2} = \frac{1}{2y} \longrightarrow 2y = 2x^2 \longrightarrow y = x^2$

$x^4 + x^4 = \cancel{16}\ 32$

$x^4 = 16$

$x = \pm 2$, $y = \cancel{2}\ 4$

Points		Value of $f(x,y)$	
$(2,4)$	=	8	} global maxima
$(-2,4)$	=	8	
$(0, 4\sqrt{2})$	=	$4\sqrt{2}$	
$(0, -4\sqrt{2})$	=	$-4\sqrt{2}$	$\longrightarrow$ minima

c. $x^4 + y^2$ was already > 0 ~~except~~ ~~at~~ for all values of x and y. Thus the new domain is larger than the previous one and so the same minimas at $(2, 4)$ and $(-2, 4)$ hold as does the minima at $(0, -4\sqrt{2})$.

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12 a. If $f(x,y)$ is differentiable, it implies that $f(x,y)$ is continuous at $(0,0)$. Thus $f(x,y)$ and $f(x,0)$ $[3x]$ must approach the same value as they both approach $(0,0)$.

Thus, assuming $\lim_{x,y \to 0,0} \left(f(x,y) - f(x,0)\right) = 0$ we get

$$\lim_{(x,y)\to(0,0)} \frac{\left(f(x,y) - f(x,0)\right)^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{x^2+y^2} = 0.$$

Thus, the statement is true.

b. True. We know that ∇F and ∇G are normal to their level surfaces. Since C is the intersection of the surfaces, there exists, at every point P on C, tangent planes to both F_P and G_P whose normal vectors are given by ∇F_P and ∇G_P respectively. The line formed by the intersection of these tangent planes at a point is the tangent line to that point and geometrically it is perpendicular to both the normals of the planes and ~~that~~ therefore can be obtained by $\nabla F_P \times \nabla G_P$ provided they are not parallel / antiparallel as then many such tangent lines would exist.