Final Exam

Instructions:

- This is a take-home exam, open book and open notes. You have 24 hours to complete it. You have to submit your solutions **on gradescope by 8am on Monday December 14, 2020**.
- Your exam should be **hand-written** (electronic pencil on a tablet is allowed).
- Please start a **new page for each question**. When uploading to gradescope, you will be asked to indicate for each question on which page(s) it can be found.
- To keep answers consistent, if you have questions while working on this exam, you should direct them by email to me: `pspaas@math.ucla.edu`. In particular, your TA will just forward any emails he gets to me, hence there is no need to email them. *Only questions about clarification of questions will be considered, no hints will be given.*
- We refer to the course syllabus for exam policies regarding academic integrity, and want to remind you that any violations will be taken seriously. Note that you have to **copy and sign** the academic integrity statement contained in question 1. Failing to do so may result in your exam receiving a failing grade.
- As usual, you must **show all your work** to receive credit. Correct answers without justification will not be awarded any points.
- Good luck!!!

Question	Points	Score
1	6	
2	6	
3	6	
4	8	
5	5	
6	5	
7	6	

Question	Points	Score
8	4	
9	6	
10	8	
11	9	
12	6	
Total:	75	

1. (a) Copy the following statement on your solution, and sign it with your **full name**, **UID**, and **signature**.

   I certify on my honor that I have neither given nor received any help, and that I have not used any non-permitted resources, while completing this assignment.

   (b) (3 points) Let $\mathbf{u} = \langle 1, -2, 0\rangle$ and $\mathbf{v} = \langle 4, -3, 6\rangle$. Find $\mathbf{v}_{\|\mathbf{u}}$ and $\mathbf{v}_{\perp\mathbf{u}}$.

   (c) (3 points) Suppose $\mathbf{v} = \langle 2, -1, -2\rangle$, $\|\mathbf{u}\| = 3$ and $\|\mathbf{w}\| = 2$. Show that the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is less than or equal to 18. (Note: these are different vectors than the ones from part (b)).

   If you have just written down the statement in part (a) without signing it, please go back and sign it now.

2. An airplane is flying along the curve $\mathbf{r}(t) = \langle 6t^2 - t, 13t, t^3 - 2t^2 + 3t\rangle$.

   (a) (2 points) What is the speed of the airplane at time $t = 2$?

   (b) (2 points) Find an equation for the tangent line to the curve at $t = 2$.

   (c) (2 points) Is there any time t at which the plane is flying in the direction given by the vector $\langle -2, 26, 0\rangle$? Explain.

3. Consider the following:

   - the curve $\mathcal{C}$ parametrized by $\mathbf{r}(t) = \langle t^2 - t, \sin(t) + 1\rangle$,
   - a differentiable function $f(x, y)$ such that $\nabla f(0, 1) = \langle 3, -2\rangle$,
   - $x(u, v) = v^2$, $y(u, v) = u^2 + uv$.

   (a) (2 points) Is $\mathbf{r}(t)$ the arc length parametrization of $\mathcal{C}$? Explain.

   (b) (2 points) Compute $\frac{d}{dt} f(\mathbf{r}(t))\big|_{t=0}$.

   (c) (2 points) Compute $\frac{\partial f}{\partial v}$ at $(u, v) = (1, 0)$.

4. Consider the straight lines in $\mathbb{R}^3$ with parametrizations $\mathbf{r}_1(t) = \langle 1 + 2t, -3t, 2 - t\rangle$ and $\mathbf{r}_2(t) = \langle 3, -5 + t, 7 - 3t\rangle$.

   (a) (3 points) Show that these lines intersect at the point $(3, -3, 1)$, and find the angle between the lines at this intersection point.

   (b) (3 points) Find an equation of the plane that contains both lines.

   (c) (2 points) Suppose the plane you found in part (b) is the tangent plane to the graph of a function $f(x, y)$ for $(x, y) = (1, 2)$. Use this to find a good estimate of the value $f(1.1, 1.9)$.

5. For $\mathbf{r}(t) = \langle 8t, 5 - 6t, -5t^2\rangle$ it is true that

$$\mathbf{T}(t) = \frac{1}{5\sqrt{t^2 + 1}}\langle 4, -3, -5t\rangle,$$

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and

$$\mathbf{N}(t) = \frac{1}{5\sqrt{t^2+1}}\langle -4t, 3t, -5\rangle.$$

(a) (2 points) Find the acceleration $\mathbf{a}(t)$.

(b) (3 points) Remember that we can always write $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ as functions of t.

6. Let S be the surface in $\mathbb{R}^3$ given by the equation $x^4 - y^3 + z^2 = 8 + 2z$.

   (a) (2 points) Find $\dfrac{\partial z}{\partial y}$ at the point $(2, 2, 0)$.

   (b) (3 points) Find all points on S where the tangent plane to S is parallel to the xy-plane.

7. For each of the following limits, calculate it, or show it doesn't exist.

   (a) (3 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2y^4}{x^4+y^8}$.

   (b) (3 points) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{y^2-4y+x+3}{(x-1)^2+(y-2)^2}$.

8. Consider the function $f(x,y)$ defined by $f(x,y) = \dfrac{3x^3-2y^3}{x^2+3y^2}$ if $(x,y) \neq (0,0)$, and $f(0,0) = 0$.

   (a) (2 points) Use the limit definition to find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

   It is a fact that $D_{\mathbf{u}}f(0,0) = 1/\sqrt{2^5}$ for $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$.

   (b) (2 points) Is $f(x,y)$ differentiable at $(0,0)$?

9. Consider the differentiable function $f(x,y) = 2\cos(x) - y$.

   (a) (2 points) Standing at the point $(\frac{\pi}{3}, 1)$ in the xy-plane, what is the largest value the directional derivative can attain? In which direction(s) is this value attained?

   (b) (2 points) Find a vector which is tangent to the level curve $f(x,y) = 0$ at the point $(\frac{\pi}{3}, 1)$.

   (c) (2 points) Sketch the level curve $f(x,y) = 0$, and draw the vector you found in part (b), based at the point $(\frac{\pi}{3}, 1)$.

10. Consider the differentiable function $f(x,y) = y^2 - 4y + x^2y$.

    (a) (6 points) Find all critical points of $f(x,y)$ and classify them (i.e. for each of them explain whether it is either a local minimum, local maximum, or saddle point).

    (b) (2 points) Does $f(x,y)$ have a global maximum when restricted to the y-axis? Explain.

11. Consider the function $f(x,y) = x^2 + y$ and the domain $D = \{(x,y) \in \mathbb{R}^2 : x^4 + y^2 \leq 32\}$.

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(a) (2 points) Give an equation for the boundary of D. Explain why it is guaranteed that $f(x, y)$ has a global maximum and minimum on D.

(b) (6 points) Find the global maximum and minimum of $f(x, y)$ on D.

(c) (1 point) Does $f(x, y)$ still have a global maximum and/or minimum on the domain $\{(x, y) \in \mathbb{R}^2 : 0 < x^4 + y^2 \leq 32\}$? Explain.

12. (a) (3 points) Suppose $f(x, y)$ is a differentiable function such that $f(x, 0) = 3x$ for all x and $f(0, y) = y^2$ for all y. True or false:

$$\lim_{(x,y)\to(0,0)} \frac{(f(x, y) - 3x)^2}{x^2 + y^2} = 0.$$

Explain.

(b) (3 points) Let $F(x, y, z)$ and $G(x, y, z)$ be differentiable functions. Suppose that the intersection of the two surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ is a curve $\mathcal{C}$, and let P be a point on $\mathcal{C}$ such that $\nabla F_P \times \nabla G_P \neq \vec{0}$. True or false: the vector $\nabla F_P \times \nabla G_P$ is a direction vector for the tangent line to $\mathcal{C}$ at P. Explain.