Midterm 2
Calculus of Several Variables
(Math 32A-002)

Name: Yijing Zhou 1E U ID: 404786683.

Question:	1	2	3	4	5	Total
Points:	20	25	25	15	15	100
Score:	5	25	24	15	15	84

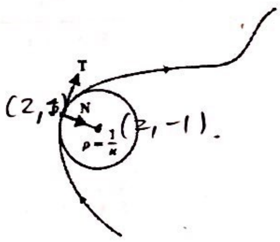

Figure 1: Osculating Circle

1. Consider the following parametrization of a curve:

$$x = \frac{1+t}{\sqrt{1+t^2}}, \quad y = \frac{1-t}{\sqrt{1+t^2}}.$$

(a) $\boxed{\text{10 points}}$ Find a relation between x and y by eliminating t.

(b) $\boxed{\text{10 points}}$ Find $\frac{dy}{dx}$ in terms of x and y only (your answer can not contain t).

[**Hint:** For Part (b), differentiate the equation in x-y obtained in Part (a) with respect to x.]

a) $x^2 = \frac{(1+t)^2}{1+t^2}$ $\quad y^2 = \frac{(1-t)^2}{1+t^2}$

$= \frac{t^2+2t+1}{1+t^2}$ $\quad = \frac{t^2-2t+1}{1+t^2}$.

$y^2 = \frac{t^2+2t+1}{1+t^2} + \frac{-4t}{1+t^2}$

$y^2 = x^2 - \frac{4t}{1+t^2}$.

$x-y = \frac{1+t-1+t}{\sqrt{1+t^2}} = \frac{2t}{\sqrt{1+t^2}}$.

$y^2 = x^2 - (x-y)2$

$y^2 = x^2 - 2(x-y)$ #

$x = \frac{1+t}{\sqrt{1+t^2}} \cdot \frac{\sqrt{1-t^2}}{\sqrt{1-t^2}}$

$= \frac{(1+t)\sqrt{1-t^2}}{1-t^2}$

$y = \frac{(1-t)\sqrt{1-t^2}}{1-t^2}$

$x = 1+t \quad y = 1-t$.

$t = 1-x$ $\quad y = 1-(1-x)$

$y = 1-1+x$

$y = \frac{x}{\sqrt{1+x^2}}$ #

b) $\frac{dy}{dx} = \frac{1+x^2}{1+x^2}$

b) $2y\,dy = (2x-2)\,dx$

$\frac{dy}{dx} = \frac{2x-2}{2y} = \frac{x-1}{y}$.

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2. The equation of the osculating circle to a curve parametrized by $\vec{r}(t) = \langle f(t), g(t)\rangle$ at the point $(2, 1)$ is:

$$x^2 + y^2 - 4x + 2y + 1 = 0.$$

(a) $\boxed{\text{5 points}}$ Find the curvature of $\vec{r}(t)$ at the point $(2, 1)$.

(b) $\boxed{\text{10 points}}$ Find the unit normal $\vec{N}$ to $\vec{r}(t)$ at the point $(2, 1)$.

(c) $\boxed{\text{10 points}}$ Find a parametrization of the tangent line to $\vec{r}(t)$ at the point $(2, 1)$. (This is not the same thing as finding the unit tangent vector $\vec{T}$ at $(2, 1)$, you need to find a parametrization of the whole tangent line).

[**Hint:** First rewrite the equation as $(x-a)^2 + (y-b)^2 = r^2$. Then look at the Figure 1 on Page 1 and use it's geometry (the relation of the osculating circle to the curve at the point $(2, 1)$) to solve this problem.]

a) $k = \dfrac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}''(t)\|}$.

$x^2 - 4x + y^2 + 2y + 1 = 0.$

$(x-2)^2 + (y+1)^2 - 4 - 1 + 1 = 0.$

$(x-2)^2 + (y+1)^2 = 4.$

$(x-2)^2 + (y+1)^2 = 2^2. \Rightarrow \vec{r}(t) = \langle 2\cos t + 2, 2\sin t - 1\rangle$

$\vec{r}'(t) = \langle -2\sin t, 2\cos t\rangle$ $\qquad \vec{r}'(t)\cdot\vec{r}''(t) = 4\sin t\cos t - 4\sin t\cos t$

$\vec{r}''(t) = \langle -2\cos t, -2\sin t\rangle.$ $\qquad = 0$

$\therefore \|\vec{r}'(t) \times \vec{r}''(t)\| = \|\vec{r}'(t)\| \cdot \|\vec{r}''(t)\| \sin(\frac{\pi}{2})$

$= \sqrt{4\sin^2 t + 4\cos^2 t} \cdot \sqrt{4\cos^2 t + 4\sin^2 t} = 2 \cdot 2 = 4.$

$k = \dfrac{4}{\|\vec{r}'(t)\|^3} = \dfrac{4}{(\sqrt{4\sin^2 t + 4\cos^2 t})^3} = \dfrac{4}{2^3} = \dfrac{4}{8} = \dfrac{1}{2}$ #. or $\frac{1}{R}$ ... #.

b). $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \dfrac{\langle -2\sin t, 2\cos t\rangle}{2} = \langle -\sin t, \cos t\rangle.$

$\vec{T}'(t) = \langle -\cos t, -\sin t\rangle.$

$\vec{N}(t) = \langle -\cos t, -\sin t\rangle \because \sqrt{\cos^2 t + \sin^2 t} = 1$ unit

$\Rightarrow x = 2 \Rightarrow 2\cos t + 2 = 2$ $\qquad 2\sin t - 1 = 1$ $\qquad \Rightarrow \vec{N}(\frac{\pi}{2}) = \langle 0, -1\rangle$ #.

$\cos t = 0$ $\qquad 2\sin t = 2$

$t = \frac{\pi}{2}.$ $\qquad$ ~~sin~~ $t = \frac{\pi}{2}$

c) $(x-2)^2+(y+1)^2 = 4.$

parametrization of tangent line

$\Rightarrow \vec{r}_t(t) = \vec{r}(t_0) + \vec{v}t.$

$\vec{v}$ = direction vector.

$\vec{v} \parallel \vec{T} = \langle -\sin t, \cos t \rangle$

at $P = (2,1)$

$\vec{T} = \langle -\sin(\frac{\pi}{2}), \cos(\frac{\pi}{2}) \rangle$

$= \langle -1, 0 \rangle.$

$\vec{v} = \lambda \langle -1, 0 \rangle$

$\vec{r}(t_0)$ can be determined by P as $P = (2,1)$ is on the curve and the circle.

$\vec{r}(t_0) = \langle 2\cos(\frac{\pi}{2})+2, 2\sin\frac{\pi}{2} - 1 \rangle.$

$= \langle 2, 1 \rangle.$

$\Rightarrow \vec{r}_t(t) = \langle 2,1 \rangle + \langle -1,0 \rangle t$

assuming $\lambda = 1$.

3. (a) $\boxed{\text{15 points}}$ Using the Squeeze Theorem prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^4+y^2}} = 0.$

(b) $\boxed{\text{10 points}}$ Use Part (a) to show that the function f defined by

$$f(x,y) = \begin{cases} \frac{x^4y^2}{\sqrt{x^4+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{at } (0,0) \end{cases}$$

is continuous at $(0,0)$.

[**Hint:** A function f is called continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.]

A). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^4+y^2}} = 0.$

must be non-negative. or zero.

$0 \le |x|^2 = x^2.$ $\quad 0 \le |y|^2 = y^2$

$0 \le |x|^4 \le x^4$ $\quad 0 \le |y|^2 \le y^2 + x^4.$ $\quad x^4 \geq 0.$

$0 \le |x|^4 \le x^4 + y^2$ $\quad y^2 \ge 0.$ $0 \le |y| \le \sqrt{x^4+y^2}.$

$0 \le |x|^2 \le \sqrt{x^4+y^2}.$

$0 \le x^2 \le \sqrt{x^4+y^2}.$

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$\therefore \Rightarrow 0 \le x^2|y| \le \left(\sqrt{x^4+y^2}\right)^2.$

$$0 \le \frac{x^2|y|}{\sqrt{x^4+y^2}} \le \sqrt{x^4+y^2}.$$

$\therefore \displaystyle\lim_{(x,y)\to(0,0)} 0 = 0 = \lim_{(x,y)\to(0,0)} \sqrt{x^4+y^2}.$

$\therefore \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2|y|}{\sqrt{x^4+y^2}} = \lim_{(x,y)\to(0,0)} \frac{(|x|)^2|y|}{\sqrt{|x|^4+|y|^2}} = 0.$

$\therefore$ since its absolute value approaches to 0 as $x,y \to 0,0$

$\Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{\sqrt{x^4+y^2}} = 0$ # shown.

b). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{\sqrt{x^4+y^2}} = \lim_{(x,y)\to(0,0)} \frac{(x^2y)^2}{\sqrt{x^4+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^4+y^2}} \cdot \lim_{(x,y)\to(0,0)} \frac{x^2y}{\cancel{\sqrt{x^4+y^2}}}$

$= 0 \cdot 0 = 0.$

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so $\therefore \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{\sqrt{x^4+y^2}} = 0 = f(0,0)$

$\therefore$ function f is continuous at $(0,0)$.

4. $\boxed{\text{15 points}}$ Compute f_{xyxzy}, where

$$f(x,y,z) = y\sin(xz)\sin(x+z) + (x+z^2)\tan y + x\tan\left(\frac{z+z^{-1}}{y-y^{-1}}\right).$$

$$f_x = y\left[z\cos(xz)\sin(x+z) + \cos(x+z)\sin(xz)\right] + \tan y + \tan\left(\frac{z+z^{-1}}{y-y^{-1}}\right)$$

$$f_{xx} = y\left[z\left(-z\sin(xz)\sin(x+z) - \cancel{\cos(xz)\cos(x+z)}\right) - \sin(x+z)\sin(xz) + \cancel{z\cos(x+z)\cos(xz)}\right]$$

$$= y\left[-z^2\sin(xz)\sin(x+z) - \sin(x+z)\sin(xz)\right]$$

$$= y(-z^2-1)\sin(xz)\sin(x+z).$$

$$f_{xxy} = (-z^2-1)\sin(xz)\sin(x+z).$$

$$f_{xxyy} = 0.$$

$$f_{xxyyz} = 0$$

$$\therefore f_{xyxzy} = f_{xxyyz} = 0.$$

5. $\boxed{\text{15 points}}$ Let L be the tangent plane to the graph of $z = f(x, y)$ at $(-2, 7, 3)$, such that it is also the tangent plane to graph of $h(x, y) = x^2 - y$ at $(-1, 1, 0)$, i.e., L is a common tangent plane to the both of the graphs. Find the equation of L.

$h(x,y) = x^2 - y$

$h_x(x,y) = 2x \qquad f_x(-1,-1) = -2.$

$h_y(x,y) = -1. \qquad h_y(-1,-1) = -1.$

$h(-1,-1) = 1-1 = 0.$

$\cancel{k}z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$

$\cancel{k}z = 0 + (-2)(x+1) - 1(y-1)$

$\cancel{k}z = -2x - 2 - y + 1.$

$-2x - y - kz - 1 = 0$

$2x + y + kz = -1$

knowing $(-2,7,3)$ is also on $2x+y+kz = -1.$

$\Rightarrow -4 + 7 + 3k = -1$

$\Rightarrow \quad 3 + 3k = -1.$

$3k = -4$

$k = -\frac{4}{3}$

$\Rightarrow L: \quad 2x + y - \frac{4}{3} = -1.$

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You don't need $z = f(x,y)$ ~~...~~ at $(-2,7,3)$ to find the equation.