Math 32A
Winter 2017
Midterm 1
1/30/2017

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This exam contains 7 pages (including this page) and 5 questions. Total of points is 100. Work neatly and show all your work, including all intermediate steps and also explain in words how you are solving a problem. Partial credits are available for most problems. Calculators are not allowed in this exam. You have 50 minutes.

Grade Table (for instructor use only)

Question	Points	Score
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
Total:	100	100

1. (20 points) Let $\vec{v}$ be the vector $(1, 2, 3)$ and $\vec{w}$ be the vector $(1, 1, 0)$
   (a) (6 points) Find the unit vector in the same direction as $\vec{v}$;
   (b) (6 points) Find the angle between the vectors $\vec{v}$ and $\vec{w}$;
   (c) (8 points) Let l_1 be the line through $(0, 0, 0)$ in the direction of $\vec{v}$ and l_2 be the line through $(-1, 0, 3)$ in the direction of $\vec{w}$. Write down the parametric equations of l_1 and l_2. Then determine if these two lines are intersecting.

a) $\vec{e}_v = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = \dfrac{(1,2,3)}{\sqrt{14}} = \boxed{\left(\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right)}$

b) $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$ $\quad \theta = \arccos\left(\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos\left(\dfrac{1+2+0}{\sqrt{14}\cdot\sqrt{2}}\right)$

$= \boxed{\arccos\left(\dfrac{3}{2\sqrt{7}}\right)}$

c) $l_1 = (0,0,0) + t(1,2,3)$ $\quad l_2 = (-1,0,3) + t(1,1,0)$

$t_1 = -1 + t_2 \longrightarrow 1 = -1 + 2$

$\left.\begin{matrix} 2t_1 = t_2 \\ 3t_1 = 3 \end{matrix}\right\} \begin{matrix} t_1 = 1 \\ t_2 = 2 \end{matrix}$ $\quad 1 \stackrel{\checkmark}{=} 1$

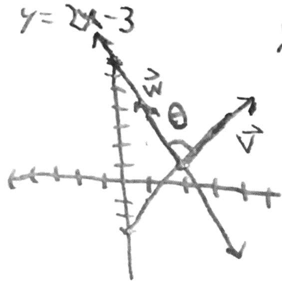
The lines intersect at $(1,2,3)$

2. (20 points) (a) (10 points) Find the acute angle between the lines $2x - y = 3$ and $3x + y = 7$.

(b) (10 points) Under what conditions is the cross product of two nonzero vectors $\vec{v}$ and $\vec{w}$ equal to the zero vector, i.e., when is $\vec{v} \times \vec{w} = \vec{0}$ where $\vec{v}, \vec{w} \neq \vec{0}$.

a) $2x - y = 3$ $\quad$ $3x + y = 7$

$y = 2x - 3$ $\quad$ $y = -3x + 7$



$l_1 = (0,-3) + t(1,2)$ $\quad$ $\vec{v} = (1,2)$

$l_2 = (0,7) + t(-1,3)$ $\quad$ $\vec{w} = (-1,3)$

Vectors representing directions of lines

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{-1+6}{\sqrt{5}\sqrt{10}} = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \boxed{\frac{\pi}{4}}$$

b) $\vec{v} \times \vec{w} = \vec{0}$ when $\vec{v} \parallel \vec{w}$, so $\vec{v} = \lambda\vec{w}$

Proof:

if $\vec{v} \parallel \vec{w}$ then the angle θ between $\vec{v}$ and $\vec{w}$ $\theta = 0$ or $\theta = \pi$

$\|\vec{v}\times\vec{w}\| = 0$ iff $\vec{v}\times\vec{w} = \vec{0}$ $\quad$ $\sin 0 = 0$ $\quad$ $\sin\pi = 0$

$\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta = \|\vec{v}\|\|\vec{w}\|\cdot 0 = 0$

therefore, $\vec{v}\times\vec{w} = \vec{0}$ iff $\theta = 0$ or $\theta = \pi$, which means that $\vec{v}$ and $\vec{w}$ are parallel

3. (20 points) Let $\vec{u} = (0, 1, -1)$ and $\vec{v} = (2, 1, 2)$. Find a vector $\vec{w}$ such that

   (a) $\vec{w} = \lambda\vec{v}$ for some scalar λ and

   (b) $\vec{u} - \vec{w} \perp \vec{v}$.

   Draw a picture illustrating the relations between $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$\vec{w} = \lambda\vec{v} = (2\lambda, \lambda, 2\lambda)$

$\vec{u} - \vec{w} = (-2\lambda, 1-\lambda, -1-2\lambda)$

$(\vec{u} - \vec{w}) \perp \vec{v} \Rightarrow (\vec{u} - \vec{w}) \cdot \vec{v} = 0$

$2(-2\lambda) + 1(1-\lambda) + 2(-1-2\lambda) = 0$

$-4\lambda + 1 - \lambda - 2 - 4\lambda = 0$

$9\lambda = -1 \quad \lambda = \frac{-1}{9} \qquad \vec{w} = -\frac{1}{9}\vec{v}$

$\vec{w} = \boxed{\left(\frac{-2}{9}, \frac{-1}{9}, \frac{-2}{9}\right)}$

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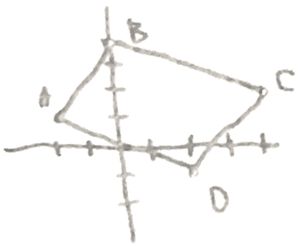
4. (20 points) Find the area of the parallelogram with vertices $A = (-2,1)$, $B = (0,4)$, $C = (4,2)$ and $D = (2,-1)$.



$$\text{Area} = \|\vec{AB} \times \vec{AD}\|$$

$$= \|(2,3,0) \times (4,-2,0)\|$$

$$= \left\| \left(\det\begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}, -\det\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}, \det\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \right) \right\|$$

$$= \|(0, 0, -4-12)\| = \sqrt{0^2+0^2+(-16)^2} = \boxed{16}$$

5. (20 points) Find the volumn of the parallelpiped spanned by $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ where the points are $A = (1,1,1)$, $B = (2,0,3)$, $C = (4,1,7)$ and $D = (3,-1,-2)$.

$\vec{AB} = (1,-1,2)$

$\vec{AC} = (3,0,6)$

$\vec{AD} = (2,-2,-3)$

$$\text{Volume} = \left|(\vec{AB} \times \vec{AC}) \cdot \vec{AD}\right|$$

$$= \left|\det\begin{pmatrix} \vec{AB} \\ \vec{AC} \\ \vec{AD} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{pmatrix}\right|$$

$$= \left|\det\begin{pmatrix} 0 & 6 \\ -2 & -3 \end{pmatrix} - (-1)\det\begin{pmatrix} 3 & 6 \\ 2 & -3 \end{pmatrix} + 2\det\begin{pmatrix} 3 & 0 \\ 2 & -2 \end{pmatrix}\right|$$

$$\left|12 - 9 - 12 + 2(-6)\right| = \left|-9-12\right| = \boxed{21}$$

$$\left|(\vec{u} \times \vec{v}) \cdot \vec{w}\right| = \left|\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\right|$$