Math 32A, Winter 2015, Midterm 1

January 28 2015

50 minutes

Name
Section (cross one):

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This exam consists of six problems, not arranged in any particular order. Please solve all problems in the space provided (or attaching additional sheets as necessary), showing all work as neatly and cleanly as possible. No calculators allowed. One sided "cheat sheet" allowed.

Determinants

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc, \qquad \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 \det \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - b_1 \det \begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} + c_1 \det \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Dot product and cross product

$$\langle a_1, b_1, c_1\rangle \cdot \langle a_2, b_2, c_2\rangle = a_1a_2+b_1b_2+c_1c_2, \qquad \langle a_1, b_1, c_1\rangle \times \langle a_2, b_2, c_3\rangle = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

Problem	Value	Score
Problem 1	15	15
Problem 2	20	20
Problem 3	20	17
Problem 4	15	15
Problem 5	20	19
Problem 6	10	10.
Total	100	96

Problem 1. (10+5 points) *Let* $\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$ *for* $t \in \mathbb{R}$.

(a) Find the projection $\mathbf{r}_{||}(t)$ *of* $\mathbf{r}(t)$ *along* $\mathbf{v} = \langle 1, 1\rangle$ *as a vector-valued function of* t.

$$r_{||}(t) = (\vec{r}(t) \cdot e_v)\, e_v \qquad e_v = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1,1\rangle}{\sqrt{1^2+1^2}} = \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right\rangle$$

$$r_{||}(t) = \left(\langle t\cos(t),\ t\sin(t)\rangle \cdot \left\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right\rangle\right)\left\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right\rangle$$

$$r_{||}(t) = \left(\frac{t\cos(t)}{\sqrt{2}} + \frac{t\sin(t)}{\sqrt{2}}\right)\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$$

$$r_{||}(t) = \left\langle \frac{t\cos(t)}{2} + \frac{t\sin(t)}{2},\ \frac{t\cos(t)}{2} + \frac{t\sin(t)}{2}\right\rangle$$

(b) Find the derivative $\frac{d}{dt}\mathbf{r}_{||}(t)$.

$$\frac{d}{dt} r_{||}(t) = \left\langle -\frac{t\sin(t)}{2} + \frac{\cos(t)}{2} + \frac{t\cos(t)}{2} + \frac{\sin(t)}{2},\right.$$

$$\left. -\frac{t\sin(t)}{2}\quad \frac{t\cos(t)}{2} + \frac{t\cos t}{2} + \frac{\sin(t)}{2}\right\rangle$$

$$= \left\langle \sin(t)\left(-\frac{t}{2} + \frac{1}{2}\right) + \cos(t)\left(\frac{1}{2} + \frac{t}{2}\right),\right.$$

$$\left. \sin(t)\left(-\frac{t}{2} + \frac{1}{2}\right) + \cos(t)\left(\frac{1}{2} + \frac{t}{2}\right)\right\rangle$$

Problem 2. (6+6+8 points) *Let $\mathcal{L}_1$ be the line line through $(-1, 0, 2)$ along the direction $\mathbf{v} = \langle 3, 1, -1 \rangle$.*

(a) *Find the equation of the line $\mathcal{L}_1$.*

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$\vec{L_1}(t) = \langle -1, 0, 2 \rangle + t \langle 3, 1, -1 \rangle$

$= \langle -1+3t, t, 2-t \rangle$

(b) *Find the intersection of $\mathcal{L}_1$ with the line $\mathcal{L}_2$ given by the equation $\mathbf{r}_2(s) = \langle 2s, s, s \rangle$.*

6

$r_2(t) = \langle 2t, t, t \rangle$

$\langle -1+3t, t, 2-t \rangle = \langle 2t, t, t \rangle$

$2 - t = t$

$2 = 2t$

$t = 1$

Point of intersection $= \langle 2, 1, 1 \rangle$

(c) *Find the equation of the plane containing the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.*

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$r_2(0) = \langle 0, 0, 0 \rangle$ $\quad P \langle 0, 0, 0 \rangle$

point of intersection $\langle 2, 1, 1 \rangle$ $\quad Q \langle 2, 1, 1 \rangle$

$\vec{L_1}(0) = \langle -1, 0, 2 \rangle$ $\quad R \langle -1, 0, 2 \rangle$

$\vec{QP} = \langle 2, 1, 1 \rangle$

$\vec{RP} = \langle -1, 0, 2 \rangle$

$\langle 2, 1, 1 \rangle \times \langle -1, 0, 2 \rangle$

$= \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix}$

$= \hat{\imath} \det \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - \hat{\jmath} \det \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$

$+ \hat{k} \det \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}$

$= 2\hat{\imath} - 5\hat{\jmath} + \hat{k}$

$= \langle 2, -5, 1 \rangle$

$\langle 0, 0, 0 \rangle \cdot \langle 2, -5, 1 \rangle$

$2x - 5y + z = 0$

Problem 3. (10+10 points) **v** *and* **w** *are two vectors of length* 4 *and* 5 *respectively in the yz-plane. The angle* θ *between them may vary.*

(a) *What is the maximum value of* $\mathbf{v} \cdot \mathbf{w}$.

$$\vec{v} \cdot \vec{w} = \|v\| \, \|w\| \cos\theta$$

$$= 4 \cdot 5 \cos\theta = 20 \cos\theta$$

-1 is the lowest value for $\cos\theta$ (at π)

$$= 20(-1) = -20$$

min !

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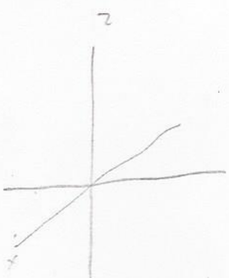
(b) *Find a vector* $\mathbf{v} \times \mathbf{w}$ *with the maximum length* $\|\mathbf{v} \times \mathbf{w}\|$.

$$\|v \times w\| = \|v\| \|w\| \sin\theta$$

$\sin\theta$ can have the maximum value of 1 (at $\frac{\pi}{2}$)

$$\|v \times w\| = 4 \cdot 5 \cdot 1 = 20$$



$$\vec{v} \to \langle 0, 0, 4 \rangle$$
$$\vec{w} \to \langle 0, 5, 0 \rangle \qquad \theta = \frac{\pi}{2}$$

$$\vec{v} \times \vec{w} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 4 \\ 0 & 5 & 0 \end{vmatrix} = \hat{i} \det \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} - \hat{j} \det \begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix}$$

$$+ \hat{k} \det \begin{vmatrix} 0 & 0 \\ 0 & 5 \end{vmatrix}$$

$$= -20\hat{i} - 0\hat{j} + 0\hat{k}$$

$$= \boxed{\langle -20, 0, 0 \rangle}$$

10

Problem 4. (15 points)
Find a parametrization for the tangent line of $\mathbf{r}(t) = \langle e^{2t}, (1+t)^2, \sin t\rangle$ *at* $t = 0$.

$r(0) = \langle e^0, (1+0)^2, \sin 0\rangle = \langle 1, 1, 0\rangle$

$r'(t) = \langle 2e^{2t}, 2(1+t), \cos t\rangle$

$r'(0) = \langle 2e^0, 2(1+0), \cos 0\rangle = \langle 2, 2, 1\rangle$

$\vec{l}(t) = \langle 1, 1, 0\rangle + t\langle 2, 2, 1\rangle$

$= \langle 2t+1, 2t+1, t\rangle$

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Problem 5. (6+14 points) *Consider the vector valued function* $\mathbf{r}(t) = \langle t^2, 1-t, e^t \rangle$.

(a) *Find* $\mathbf{r}'(t) = \frac{d}{dt}\mathbf{r}(t)$ *and* $\mathbf{r}''(t) = \frac{d^2}{dt^2}\mathbf{r}(t)$.

$r'(t) = \langle 2t, -1, et \rangle$ ✓

6 $r''(t) = \langle 2, 0, e^t \rangle$

(b) *Find a value* $t = t_0$ *such that the volume of the parallelepiped* $\mathcal{P}(t_0)$ *spanned by the vectors* $\mathbf{r}(t_0), \mathbf{r}'(t_0)$, *and* $\mathbf{r}''(t_0)$ *has volume* 4.

$$4 = \text{volume} = \left| \det \begin{vmatrix} t_0^2 & 1-t_0 & e^{t_0} \\ 2t_0 & -1 & e^{t_0} \\ 2 & 0 & e^{t_0} \end{vmatrix} \right| = t^2 \det \begin{vmatrix} -1 & e^{t_0} \\ 0 & e^{t_0} \end{vmatrix} - (1-t) \det \begin{vmatrix} 2t_0 & e \\ 2 & e \end{vmatrix}$$

$$+ e^t \det \begin{vmatrix} 2t_0 & -1 \\ 2 & 0 \end{vmatrix}$$

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$$= t_0^2(-e^{t_0}) - (1-t_0)(2t_0e^{t_0} - 2e^{t_0}) + e^{t_0}(2)$$

(−1)

$$= \left| -e^{t_0}t_0^2 + (1-t_0)(2e^{t_0} - 2t_0e^{t_0}) + 2e^{t_0} \right|.$$

$$= -t_0^2e^{t_0} + 2e^{t_0} - 2e^{t_0}t_0 - 2t_0e^{t_0} + 2e^{t_0}t_0^2 + 2e^{t_0}$$

$$= e^{t_0}t_0^2 + 4e^{t_0} - 4e^{t_0}t_0$$

$$= e^{t_0}(t_0^2 + 4 - 4t_0) = 4$$

$t_0 = 0$

Problem 6. (10 points) *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *be nonzero vectors in three dimensions, prove that*

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})| = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|.$$

10

$$\cancel{|(u \cdot v) \times (u \cdot w)| = |(w \cdot v) \times (u \cdot v)|} = |(w \cdot u) \times (w \cdot v$$

$$\left| \det \begin{vmatrix} u \\ v \\ w \end{vmatrix} \right| = \left| \det \begin{vmatrix} v \\ w \\ u \end{vmatrix} \right| = \left| \det \begin{vmatrix} w \\ u \\ v \end{vmatrix} \right|$$

$$\left| - \det \begin{vmatrix} u \\ w \\ v \end{vmatrix} \right| = \left| - \det \begin{vmatrix} u \\ w \\ v \end{vmatrix} \right| = \left| - \det \begin{vmatrix} u \\ w \\ v \end{vmatrix} \right| \checkmark$$

$$|(\vec{v} \times \vec{w}) \vec{u}| = \det \begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix}$$

$$|u \cdot (v \times w)| = \det \begin{vmatrix} u \\ v \\ w \end{vmatrix}$$

$$|v \cdot (w \times u)| = \det \begin{vmatrix} v \\ w \\ u \end{vmatrix} = - \det \begin{vmatrix} u \\ w \\ v \end{vmatrix} = \det \begin{vmatrix} u \\ v \\ w \end{vmatrix}$$

$$|w \cdot (u \times v)| = \det \begin{vmatrix} w \\ u \\ v \end{vmatrix} = - \det \begin{vmatrix} u \\ w \\ v \end{vmatrix} = \det \begin{vmatrix} u \\ v \\ w \end{vmatrix}$$

thus $|u \cdot (v \times w)| = |v \cdot (w \times u)| = |w \cdot (u \times v)|$ $\checkmark$