20S-MATH32A-3 Midterm 2

SAMUEL ALSUP

TOTAL POINTS

89 / 100

QUESTION 1

Frenet Formulas 25 pts

1.1 Arc length function 10 / 10

✓ + **2 pts** Correct velocity r'(t) = <-5sin(t), -4cos(t), 3cos(t) >

✓ + **2 pts** Correct speed ||r(t)|| = 5

✓ + **2 pts** Correct arc length integral s = int_0^t (speed) d tau. Full points even if the speed is incorrect.

✓ + **4 pts** Obtains s = 5t

+ **1 pts** Has incorrect bounds in the arc length integral.

+ **2 pts** Has incorrect speed but carries out the calculation with the correct formula for arc length.

1.2 Unit Tangent 5 / 5

✓ + **2 pts** Divides r'(t) obtained in Part (a) by the speed ||r'(t)|| obtained in part (a)

✓ + **3 pts** Gets T = <-sin(t), -4/5cos(t), 3/5cos(t)>. 1 Point is awarded for each correct component. May also answer in arc length parametrization T = <-sin(s/5), -4/5cos(s/5), 3/5cos(s/5)>. Full points also awarded if r'(t) or ||r'(t)|| from Part (a) is incorrect but the procedure is correct.

+ **2 pts** 2 out of 3 components of T are correct

+ **1 pts** 1 out of 3 components is correct.

+ **0 pts** No points awarded.

1.3 Curvature and unit normal 10 / 10

✓ + **2 pts** PARTIAL CREDIT: Correctly computes dT/ds = 1/5*<-cos(t), 4/5 sin(t), -3/5 sin(t) >. May also express answer in terms of arc length parametrization: dT/ds = 1/5*<-cos(s/5), 4/5 sin(s/5), -3/5 sin(s/5)>. Full points are awarded if kappa and N are correct, but the student computed these without computing dT/ds explicitly.

✓ + **4 pts** Obtains \\kappa = 1/5

✓ + **4 pts** Computes N = <-cos(t), 4/5 sin(t), -3/5 sin(t)>. May also express answer in terms of arc length parametrization: N = <-cos(s/5), 4/5 sin(s/5), -3/5 sin(s/5)>.

+ **0 pts** No partial credit.

QUESTION 2

2 Limits 10 / 10

+ **3 pts** Made some remark that it was insufficient to test along lines, but was vague, too brief, or partially wrong

✓ + **5 pts** Stated that it was insufficient to test along lines to show that a limit exists, since the limit may be different along other paths. Also give points if student said that this method is only sufficient to determine nonexistence

✓ + **2 pts** Converted to polar

✓ + **1 pts** Correctly bounded the part of the function depending on theta

✓ + **2 pts** Applied the squeeze theorem (needed to do this to get full credit)

+ **2 pts** Wrote something correct but missed the points above

+ **0 pts** No credit (e.g. claimed limit does not exist)

QUESTION 3

Continuity 15 pts

3.1 Domain 4 / 5

+ **5 pts** Correct domain (all points EXCEPT y = 0 and x/y = pi/2 + k pi, k an integer)

✓ + **4 pts** Did not restrict enough values for tan OR restricted too many values of tan OR wrote (cos(x/y) unequal to 0) and did not get explicit domain

restriction (but otherwise correct)

- **+ 3 pts** Got tan restriction correct but forgot y unequal to 0
- **+ 2 pts** Only restricted y unequal to 0
- **+ 0 pts** No credit (e.g. did not give a set of points in R^2)

3.2 Continuous? 3 / 5

- ✓ **+ 3 pts Stated same domain as above (give credit even if answer to first question is wrong). Also give credit if domain here was correct but is wrong above. (I.e. this part is for establishing where function is undefined and thus not continuous)**
- **+ 2 pts** Explained that the function is a composition of continuous functions and/or that both x/y and tan( ) are continuous where defined (must argue these somehow, otherwise you're just asserting that the function as a whole is continuous without justification) To get this part had to mention that x/y IS continuous where defined.
- **+ 0 pts** No credit

3.3 Limit 5 / 5

- ✓ **+ 5 pts =1, student appealed to continuity, or at least implied this by showing (pi,4) belonged to the set given in (b)**
- **+ 4 pts** =1. student's justification was unclear, maybe just saying that the point belonged to the domain (but not explicitly saying that the function was continuous at that point)
- **+ 3 pts** =1, student did not give correct justification
- **+ 2 pts** Student appealed to continuity but got the answer wrong or did not fully solve the limit
- **+ 0 pts** No credit

QUESTION 4

Partial and directional derivatives 25 pts

4.1 Partials 5 / 5

- ✓ **+ 2.5 pts $$\frac{\partial f}{\partial x} = 3x^2 - 3$$**
- ✓ **+ 2.5 pts $$\frac{\partial f}{\partial y} = -6y^2 + 6y$$**
- **- 1 pts** Small algebra mistake
- **- 2 pts** Two small algebra mistakes

4.2 Gradient 5 / 5

- ✓ **+ 2.5 pts $$\nabla f = \langle \frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} \rangle $$**
- ✓ **+ 2.5 pts $$\nabla f = \langle -3, -12 \rangle $$, or a correct gradient given your computations from (a)**
- **- 1 pts** Small algebra mistake
- **- 2 pts** Two small algebra mistakes

4.3 Directional derivative 5 / 5

- ✓ **+ 2 pts Computed unit vector $$\vec{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$$ in the direction of $$\vec{v}$$**
- ✓ **+ 2 pts $$D_{\vec{v}}f(0,-1) = \nabla f_{(0,-1)} \cdot \vec{u}$$**
- ✓ **+ 1 pts $$D_{\vec{v}}f(0,-1) = \frac{-57}{5}$$, or correct given previous computations**
- **- 1 pts** Small algebra mistake
- **- 2 pts** Two small algebra mistakes
- **+ 3 pts** (partial) Dotted the gradient with $$\vec{v}$$ instead of with $$\vec{u}$$

4.4 Isocline 5 / 5

- ✓ **+ 3 pts Must walk perpendicular to the gradient**
- ✓ **+ 1 pts Direction vector will satisfy $$\langle v_1, v_2 \rangle \cdot \langle -3, -12\rangle = 0$$**
- ✓ **+ 1 pts Any nonzero vector of the form $$\langle -4c, c\rangle$$ works.**
- **+ 0 pts** Blank or incorrect

4.5 Max slope 2 / 5

- **+ 2 pts** Maximum value of the direction derivative is the magnitude of the gradient
- **+ 2 pts** At $$(0,-1)$$, $$||\vec{\nabla{f}}|| = \sqrt{153}$$
- **+ 1 pts** $$\sqrt{153} < 13$$, so the directional derivative cannot be greater than $$13$$
- **- 1 pts** Small algebra mistake
- **+ 0 pts** Blank or incorrect
- ✓ **+ 2 pts (partial) Correct answer but no justification or incorrect justification.**

The direction derivative can be greater than 12.

QUESTION 5

Contour 25 pts

5.1 Gradients 6 / 10

+ **10 pts** Correct
+ **2 pts** One perpendicular

✓ + **4 pts** **Both perpendicular**

✓ + **2 pts** **One points in right direction**

+ **4 pts** Both point in right direction
+ **2 pts** Lengths scaled

- **1 pts** Drawing mistake

+ **0 pts** Incorrect or unclear

5.2 Reasoning 4 / 5

✓ + **5 pts** **Correct**

✓ - **1 pts** **Length argument wrong**

+ **2 pts** Recognized role of steepest ascent
+ **1 pts** Only addressed length
+ **0 pts** Not correct
+ **2 pts** Recognizes perpendicular to level curve

- **2 pts** Wrong wording for directions

5.3 Geometry 5 / 5

✓ + **5 pts** **Correct**

+ **0 pts** Wrong geometry

5.4 Near 1.5, 1.5 5 / 5

✓ + **5 pts** **Correct**

+ **0 pts** Wrong geometry

MATH 32A MIDTERM #2 MAY 13, 2020

NAME: Samuel Alsup ID: 805 371 633

SIGNATURE:

To get credit for a problem, you must show all of your reasoning and calculations. You may consult your books, notes, calculator, any materials from the CCLE site, the professor, or your TA. You may not collaborate or ask questions online. $\boxed{\textit{Box your final answer.}}$
If you cannot find a vector that you need for a later part of a problem, you may use the vector $\langle 1, 2, 3\rangle$.
If you cannot find a point that you need for a later part of a problem, you may use the point $(1, 1, 1)$.

Circle your section:

Section:	**Tuesday:**	**Thursday:**	**TA:**
	2A	*2B*	*Alexander Johnson*
	2C	[✓ *2D*]	*Francis White*
	2E	*2F*	*Jason Snyder*

Problem	**Possible**	**Points**
1	*25*	
2	*10*	
3	*15*	
4	*25*	
5	*25*	
Total	*100*	

1. *(25 points)*

$$\vec{r}(t) = \langle 5\cos(t), -4\sin(t), 3\sin(t)\rangle.$$

(a) (10 points) Find the arc length starting from $t = 0$ as a function of t.

$\vec{r}'(t) = \langle -5\sin(t), -4\cos(t), 3\cos(t)\rangle \rightarrow \|\vec{r}'(t)\| = \sqrt{(-5\sin(t))^2 + (-4\cos(t))^2 + (3\cos(t))^2}$

$\vee$

$\sqrt{25\sin^2(t) + 16\cos^2(t) + 9\cos^2(t)}$

$\vee$

$\sqrt{25\sin^2(t) + 25\cos^2(t)}$

$25\sin^2(t) + 25\cos^2(t) = \sqrt{\boxed{25}}$

$= 5$

$\int_0^t 5\,dt = 5t\Big|_0^t = \boxed{5t}$

1.1 Arc length function 10 / 10

- ✓ **+ 2 pts** **Correct velocity r'(t) = <-5sin(t), -4cos(t), 3cos(t) >**
- ✓ **+ 2 pts** **Correct speed ||r(t)|| = 5**
- ✓ **+ 2 pts** **Correct arc length integral s = int_0^t (speed) d tau. Full points even if the speed is incorrect.**
- ✓ **+ 4 pts** **Obtains s = 5t**
- **+ 1 pts** Has incorrect bounds in the arc length integral.
- **+ 2 pts** Has incorrect speed but carries out the calculation with the correct formula for arc length.

(b) (5 points) Find the unit tangent $\vec{T}$

$$\vec{T} = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle -5\sin(t), -4\cos(t), 3\cos(t) \rangle}{5} =$$

$$\boxed{\langle -\sin(t), \tfrac{-4}{5}\cos(t), \tfrac{3}{5}\cos(t) \rangle}$$

(c) (10 points) Find the curvature and unit normal $\vec{N}$.

$$\kappa = \frac{\| r'(t) \times r''(t) \|}{\|r'(t)\|^3}$$

$$r''(t) = \langle -5\cos(t), 4\sin(t), -3\sin(t) \rangle$$

$$\begin{vmatrix} i & j & k \\ -5\sin t & -4\cos t & 3\cos t \\ -5\cos t & 4\sin t & -3\sin t \end{vmatrix} = i \begin{vmatrix} -4\cos t & 3\cos t \\ 4\sin t & -3\sin t \end{vmatrix} - j \begin{vmatrix} -5\sin t & 3\cos t \\ -5\cos t & -3\sin t \end{vmatrix}$$

$$+ k \begin{vmatrix} -5\sin t & -4\cos t \\ -5\cos t & 4\sin t \end{vmatrix}$$

$-(15\sin^2 t + 15\cos^2 t)$

-15

$12\cos t\sin t - 12\cos t\sin t = 0$

$-20\sin^2 t - 20\cos^2 t$

$= \langle 0, -15, -20 \rangle$

$= \sqrt{15^2+20^2} = \sqrt{225+400} = \sqrt{62}$

$\|r'(t) \times r''(t)\| \rightarrow = 25$

$$\kappa = \frac{25}{\|r'(t)\|^3} \rightarrow 5^3 = 125$$

$$\boxed{\kappa = \frac{25}{125} = \boxed{\frac{1}{5}}}$$

$$\vec{N} = \frac{T'(t)}{\|T'(t)\|} = \frac{\langle -\cos(t), \frac{4}{5}\sin(t), -\frac{3}{5}\sin(t) \rangle}{1}$$

$$\boxed{\vec{N} = \langle -\cos(t), \tfrac{4}{5}\sin(t), -\tfrac{3}{5}\sin(t) \rangle}$$

1.2 Unit Tangent **5 / 5**

- ✓ **+ 2 pts** **Divides r'(t) obtained in Part (a) by the speed ||r'(t)|| obtained in part (a)**
- ✓ **+ 3 pts** **Gets T = <-sin(t), -4/5cos(t), 3/5cos(t)>. 1 Point is awarded for each correct component. May also answer in arc length parametrization T = <-sin(s/5), -4/5cos(s/5), 3/5cos(s/5)>. Full points also awarded if r'(t) or ||r'(t)|| from Part (a) is incorrect but the procedure is correct.**
  - **+ 2 pts** 2 out of 3 components of T are correct
  - **+ 1 pts** 1 out of 3 components is correct.
  - **+ 0 pts** No points awarded.

(b) (5 points) Find the unit tangent $\vec{T}$

$$\vec{T} = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle -5\sin(t), -4\cos(t), 3\cos(t)\rangle}{5} =$$

$$\boxed{\langle -\sin(t), \frac{-4}{5}\cos(t), \frac{3}{5}\cos(t)\rangle}$$

(c) (10 points) Find the curvature and unit normal $\vec{N}$.

$$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$$r''(t) = \langle -5\cos(t), 4\sin(t), -3\sin(t)\rangle$$

$$\begin{vmatrix} i & j & k \\ -5\sin t & -4\cos t & 3\cos t \\ -5\cos t & 4\sin t & -3\sin t \end{vmatrix} = i\begin{vmatrix} -4\cos t & 3\cos t \\ 4\sin t & -3\sin t \end{vmatrix} - j\begin{vmatrix} -5\sin t & 3\cos t \\ -5\cos t & -3\sin t \end{vmatrix}$$

$$+k\begin{vmatrix} -5\sin t & -4\cos t \\ -5\cos t & 4\sin t \end{vmatrix}$$

$$-(15\sin^2 t + 15\cos^2 t)$$
$$-15$$

$$12\cos t\sin t - 12\cos t\sin t = 0$$

$$-20\sin^2 t - 20\cos^2 t$$

$$= \langle 0, -15, -20\rangle$$

$$= \sqrt{15^2+20^2} = \sqrt{225+400} = \sqrt{62$$

$$\|r'(t)\times r''(t)\| \rightarrow = 25$$

$$\kappa = \frac{25}{\|r'(t)\|^3} \rightarrow 5^3 = 125$$

$$\boxed{\kappa = \frac{25}{125} = \boxed{\frac{1}{5}}}$$

$$\vec{N} = \frac{T'(t)}{\|T'(t)\|} = \frac{\langle -\cos(t), \frac{4}{5}\sin(t), -\frac{3}{5}\sin(t)\rangle}{1}$$

$$\boxed{\vec{N} = \langle -\cos(t), \frac{4}{5}\sin(t), -\frac{3}{5}\sin(t)\rangle}$$

1.3 Curvature and unit normal **10 / 10**

- ✓ **+ 2 pts** **PARTIAL CREDIT: Correctly computes dT/ds = 1/5*<-cos(t), 4/5 sin(t), -3/5 sin(t) >. May also express answer in terms of arc length parametrization: dT/ds = 1/5*<-cos(s/5), 4/5 sin(s/5), -3/5 sin(s/5)>. Full points are awarded if kappa and N are correct, but the student computed these without computing dT/ds explicitly.**
- ✓ **+ 4 pts** **Obtains \\kappa = 1/5**
- ✓ **+ 4 pts** **Computes N = <-cos(t), 4/5 sin(t), -3/5 sin(t)>. May also express answer in terms of arc length parametrization: N = <-cos(s/5), 4/5 sin(s/5), -3/5 sin(s/5)>.**
- **+ 0 pts** No partial credit.

2. *(10 points) Snape says "We have*

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{\sqrt{x^2+y^2}} = 0$$

because when we test against lines $y = mx$, we always get 0. Therefore the limit exists and is zero." What's wrong with this argument? Please provide a correct argument.

What's wrong with this argument is that testing against the lines $y=mx$ can only prove that a limit doesn't exist. This argument cannot be used to say that a limit exists, because at a certain point, the two sides of the graph could be approaching in a spiral or curve, not necessarily a line. A correct argument can be using polar where $x = r\cos\theta$ and $y = r\sin\theta$

$$\lim_{r\to 0} \frac{(r\cos\theta)^2 + r\cos\theta\, r\sin\theta + (r\sin\theta)^2}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}} = \frac{r^2 + r\cos\theta\, r\sin\theta}{r} = \frac{r^2(1+\cos\theta\sin}{r}$$

$$\downarrow$$

$$\lim_{r\to 0} r(1+\cos\theta\sin\theta)$$

Then, we use Squeeze Theorem

max value of $\cos\theta\sin\theta$ is at $\frac{\pi}{4} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{2}{4} = \frac{1}{2}$

$$-\frac{1}{2} \le \cos\theta\sin\theta \le \frac{1}{2}$$

$$\frac{1}{2} \le \cos\theta\sin\theta + 1 \le \frac{3}{2}$$

$$\frac{r}{2} \le r(\cos\theta\sin\theta+1) \le \frac{3r}{2}$$

$$\lim_{r\to 0}\left(\frac{r}{2}\right) \le \lim_{r\to 0}\left(r(\cos\theta\sin\theta+1)\right) \le \lim_{r\to 0}\left(\frac{3r}{2}\right)$$

$\lim_{r\to 0}\left(\frac{r}{2}\right) = 0$ and $\lim_{r\to 0}\left(\frac{3r}{2}\right) = 0$, so therefore $\boxed{\lim_{r\to 0} r(\cos\theta\sin\theta+1) = \boxed{0}}$

2 Limits 10 / 10

- \+ **3 pts** Made some remark that it was insufficient to test along lines, but was vague, too brief, or partially wrong
- ✓ + **5 pts** **Stated that it was insufficient to test along lines to show that a limit exists, since the limit may be different along other paths. Also give points if student said that this method is only sufficient to determine nonexistence**
- ✓ + **2 pts** **Converted to polar**
- ✓ + **1 pts** **Correctly bounded the part of the function depending on theta**
- ✓ + **2 pts** **Applied the squeeze theorem (needed to do this to get full credit)**
- \+ **2 pts** Wrote something correct but missed the points above
- \+ **0 pts** No credit (e.g. claimed limit does not exist)

3. (15 points) Let

$$f(x, y) = \tan(x/y)$$

(a) What is the largest possible domain of $f(x, y)$? Why?

The domain of tangent is any value where $\cos(x/y)$ is not equal to 0, and y is not equal to 0. This is because if $\cos(x/y)$ is equal to 0, $\tan(x/y) = \frac{\sin(x/y)}{\cos(x/y)}$ will be dividing by 0 which is undefined. This is also the case for when y is 0, as then it will be tangent of an undefined number.

So, the domain is $\{(x,y) \in \mathbb{R}^2 \mid \cos(\frac{x}{y}) \neq 0 \text{ and } y \neq 0\}$

$$\boxed{= \{(x,y) \in \mathbb{R}^2 \mid \frac{x}{y} \neq \pm\frac{n\pi}{2} \text{ and } y \neq 0\} \text{ where } n \text{ is any integer}}$$

(b) Where is the function continuous and why?

The function is continuous everywhere $\cos(\frac{x}{y}) \neq 0$, which is anywhere where $\frac{x}{y} \neq \pm\frac{\pi}{2}$.

The other stipulation is that the function is only continuous where $y \neq 0$, so, therefore the function is continuous on $\boxed{\{(x,y) \in \mathbb{R}^2 \mid y \neq 0 \text{ and } \frac{x}{y} \neq \pm\frac{n\pi}{2}\}}$ $\boxed{\text{where } n \text{ is any integer}}$

(c) Find

$$\lim_{(x,y)\mapsto(\pi,4)} f(x, y)$$

with justification.

$f(x,y)$ is continuous at $(x,y) = (\pi, 4)$ as $4 \neq 0$ and $\frac{\pi}{4} \neq \frac{\pi}{2}$, so, we can just plug it in

$$\lim_{(x,y)\to(\pi,4)} f(x,y) = \tan\frac{\pi}{4} = \frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \boxed{1}$$

3.1 Domain 4 / 5

+ **5 pts** Correct domain (all points EXCEPT y = 0 and x/y = pi/2 + k pi, k an integer)

✓ + **4 pts** **Did not restrict enough values for tan OR restricted too many values of tan OR wrote (cos(x/y) unequal to 0) and did not get explicit domain restriction (but otherwise correct)**

+ **3 pts** Got tan restriction correct but forgot y unequal to 0

+ **2 pts** Only restricted y unequal to 0

+ **0 pts** No credit (e.g. did not give a set of points in R^2)

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So, the domain is $\{(x,y) \in \mathbb{R}^2 \mid \cos(\frac{x}{y}) \neq 0 \text{ and } y \neq 0\}$

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The function is continuous everywhere $\cos(\frac{x}{y}) \neq 0$, which is anywhere where $\frac{x}{y} \neq \pm\frac{\pi}{2}$.

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(c) Find

$$\lim_{(x,y)\mapsto(\pi,4)} f(x,y)$$

with justification.

$f(x,y)$ is continuous at $(x,y) = (\pi, 4)$ as $4 \neq 0$ and $\frac{\pi}{4} \neq \frac{\pi}{2}$, so, we can just plug it in

$$\lim_{(x,y)\to(\pi,4)} f(x,y) = \tan\frac{\pi}{4} = \frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\boxed{1}$$

3.2 Continuous? 3 / 5

✓ **+ 3 pts Stated same domain as above (give credit even if answer to first question is wrong). Also give credit if domain here was correct but is wrong above. (I.e. this part is for establishing where function is undefined and thus not continuous)**

+ 2 pts Explained that the function is a composition of continuous functions and/or that both x/y and tan( ) are continuous where defined (must argue these somehow, otherwise you're just asserting that the function as a whole is continuous without justification) To get this part had to mention that x/y IS continuous where defined.

+ 0 pts No credit

3. (15 points) Let

$$f(x, y) = \tan(x/y)$$

(a) What is the largest possible domain of $f(x, y)$? Why?

The domain of tangent is any value where $\cos(x/y)$ is not equal to 0, and y is not equal to 0. This is because if $\cos(x/y)$ is equal to 0, $\tan(x/y) = \frac{\sin(x/y)}{\cos(x/y)}$ will be dividing by 0 which is undefined. This is also the case for when y is 0, as then it will be tangent of an undefined number.

So, the domain is $\{(x,y) \in \mathbb{R}^2 \mid \cos(\frac{x}{y}) \neq 0 \text{ and } y \neq 0\}$

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The function is continuous everywhere $\cos(\frac{x}{y}) \neq 0$, which is anywhere where $\frac{x}{y} \neq \pm\frac{\pi}{2}$.

The other stipulation is that the function is only continuous where $y \neq 0$, so, therefore the function is continuous on $\boxed{\{(x,y) \in \mathbb{R}^2 \mid y \neq 0 \text{ and } \frac{x}{y} \neq \pm\frac{n\pi}{2}\}}$ $\boxed{\text{where } n \text{ is any integer}}$

(c) Find

$$\lim_{(x,y)\mapsto(\pi,4)} f(x, y)$$

with justification.

$f(x,y)$ is continuous at $(x,y) = (\pi, 4)$ as $4 \neq 0$ and $\frac{\pi}{4} \neq \frac{\pi}{2}$, so, we can just plug it in

$$\lim_{(x,y)\to(\pi,4)} f(x,y) = \tan\frac{\pi}{4} = \frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} =$$

$$\boxed{1}$$

3.3 Limit 5 / 5

- ✓ **+ 5 pts** **=1, student appealed to continuity, or at least implied this by showing (pi,4) belonged to the set given in (b)**
- **+ 4 pts** =1. student's justification was unclear, maybe just saying that the point belonged to the domain (but not explicitly saying that the function was continuous at that point)
- **+ 3 pts** =1, student did not give correct justification
- **+ 2 pts** Student appealed to continuity but got the answer wrong or did not fully solve the limit
- **+ 0 pts** No credit

4. *(25 points) Consider the function*

$$f(x, y) = x^3 - 3x - 2y^3 + 3y^2 - 1$$

(a) *Find*

$$\frac{\partial f}{\partial x} \quad \& \quad \frac{\partial f}{\partial y}.$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 3 \\ \frac{\partial f}{\partial y} &= -6y^2 + 6y \end{aligned}}$$

(b) *Find $\vec{\nabla} f$ at the point $(0, -1)$.*

$\vec{\nabla} f$ at the point $(0,-1) = \langle \frac{\partial f}{\partial x}(0,-1), \frac{\partial f}{\partial y}(0,-1) \rangle = \boxed{\langle -3, -12 \rangle}$

(c) *If $\vec{v} = \langle 3, 4 \rangle$, then find the directional derivative of f in the direction of $\vec{v}$ at the point $(0, -1)$.*

find unit vector: $||\vec{v}|| = \sqrt{3^2+4^2} = \sqrt{25} = 5$, so $\vec{v} = \vec{e}_{\vec{v}} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\vec{v}} f(0,-1) = \langle -3, -12 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = -\frac{9}{5} + \frac{-48}{5} = \boxed{\frac{-57}{5}}$

$\vec{\nabla} f_{@(0,-1)}$ unit vector in direction of $\vec{v}$

(d) *At the point $(0, -1)$, what is a direction you could begin to walk in if you want to walk along an isocline (i.e. leaving the height unchanged)?*

$\vec{\nabla} f_{(0,-1)} = \langle -3, -12 \rangle$ points in the direction of maximum rate of increase, so we need normal vector.

$\langle -3, -12 \rangle \cdot \langle -4, 1 \rangle = 12 - 12 = 0$, $\sqrt{4^2+1^2} = \sqrt{17} = \boxed{\langle \frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle}$

(e) *Are there any directions $\vec{u}$ where the directional derivative $D_{\vec{u}} f(0, -1)$ is greater than 13?*

$D_{\vec{v}} f(0,-1) = \langle -3, -12 \rangle \cdot \vec{v} > 13$

No, because the greatest number we can achieve is 12, where $\vec{v} = \langle 0, -1 \rangle$

4.1 Partials 5 / 5

- ✓ **+ 2.5 pts** **$$\frac{\partial f}{\partial x} = 3x^2 - 3$$**
- ✓ **+ 2.5 pts** **$$\frac{\partial f}{\partial y} = -6y^2 + 6y$$**
- **- 1 pts** Small algebra mistake
- **- 2 pts** Two small algebra mistakes

4\. *(25 points) Consider the function*

$$f(x, y) = x^3 - 3x - 2y^3 + 3y^2 - 1$$

(a) *Find*

$$\frac{\partial f}{\partial x} \quad \& \quad \frac{\partial f}{\partial y}.$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 3 \\ \frac{\partial f}{\partial y} &= -6y^2 + 6y \end{aligned}}$$

(b) *Find* $\vec{\nabla} f$ *at the point* $(0, -1)$.

$\vec{\nabla} f$ at the point $(0,-1) = \langle \frac{\partial f}{\partial x}(0,-1), \frac{\partial f}{\partial y}(0,-1) \rangle = \boxed{\langle -3, -12 \rangle}$

(c) *If* $\vec{v} = \langle 3, 4 \rangle$, *then find the directional derivative of* f *in the direction of* $\vec{v}$ *at the point* $(0, -1)$.

find unit vector: $\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$, so $\vec{u} = \vec{e}_{\vec{v}} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\vec{v}} f(0,-1) = \underbrace{\langle -3, -12 \rangle}_{\vec{\nabla} f_{@(0,-1)}} \cdot \underbrace{\langle \frac{3}{5}, \frac{4}{5} \rangle}_{\text{unit vector in direction of } \vec{v}} = -\frac{9}{5} + \frac{-48}{5} = \boxed{\frac{-57}{5}}$

(d) *At the point* $(0, -1)$, *what is a direction you could begin to walk in if you want to walk along an isocline (i.e. leaving the height unchanged)?*

$\vec{\nabla} f_{(0,-1)} = \langle -3, -12 \rangle$ points in the direction of maximum rate of increase, so we need normal vector.

$\langle -3, -12 \rangle \cdot \langle -4, 1 \rangle = 12 - 12 = 0$, $\sqrt{4^2+1^2} = \sqrt{17}$ $= \boxed{\langle \frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle}$

(e) *Are there any directions* $\vec{u}$ *where the directional derivative* $D_{\vec{u}} f(0, -1)$ *is greater than 13?*

$D_{\vec{v}} f(0,-1) = \langle -3, -12 \rangle \cdot \vec{v} > 13$

No, because the greatest number we can achieve is 12, where $\vec{v} = \langle 0, -1 \rangle$

4.2 Gradient **5 / 5**

- ✓ **+ 2.5 pts** **$$\nabla f = \langle \frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} \rangle $$**
- ✓ **+ 2.5 pts** **$$\nabla f = \langle -3, -12 \rangle $$, or a correct gradient given your computations from (a)**
- **- 1 pts** Small algebra mistake
- **- 2 pts** Two small algebra mistakes

4. (25 points) Consider the function

$$f(x,y) = x^3 - 3x - 2y^3 + 3y^2 - 1$$

(a) Find

$$\frac{\partial f}{\partial x} \quad \& \quad \frac{\partial f}{\partial y}.$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 3 \\ \frac{\partial f}{\partial y} &= -6y^2 + 6y \end{aligned}}$$

(b) Find $\vec{\nabla} f$ at the point $(0,-1)$.

$\vec{\nabla} f$ at the point $(0,-1) = \langle \frac{\partial f}{\partial x}(0,-1), \frac{\partial f}{\partial y}(0,-1) \rangle = \boxed{\langle -3, -12 \rangle}$

(c) If $\vec{v} = \langle 3,4 \rangle$, then find the directional derivative of f in the direction of $\vec{v}$ at the point $(0,-1)$.

find unit vector: $\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$, so $\vec{u} = \vec{e}_{\vec{v}} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\vec{v}} f(0,-1) = \underset{\vec{\nabla} f_{@(0,-1)}}{\langle -3, -12 \rangle} \cdot \underset{\text{unit vector in direction of } \vec{v}}{\langle \frac{3}{5}, \frac{4}{5} \rangle} = -\frac{9}{5} + \frac{-48}{5} = \boxed{\frac{-57}{5}}$

(d) At the point $(0,-1)$, what is a direction you could begin to walk in if you want to walk along an isocline (i.e. leaving the height unchanged)?

$\vec{\nabla} f_{(0,-1)} = \langle -3, -12 \rangle$ points in the direction of maximum rate of increase, so we need normal vector.

$\langle -3, -12 \rangle \cdot \langle -4, 1 \rangle = 12 - 12 = 0$, $\sqrt{4^2+1^2} = \sqrt{17} = \boxed{\langle \frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle}$

(e) Are there any directions $\vec{u}$ where the directional derivative $D_{\vec{u}} f(0,-1)$ is greater than 13?

$D_{\vec{v}} f(0,-1) = \langle -3, -12 \rangle \cdot \vec{v} > 13$

No, because the greatest number we can achieve is 12, where $\vec{v} = \langle 0, -1 \rangle$

4.3 Directional derivative **5 / 5**

- ✓ **+ 2 pts Computed unit vector $$\vec{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$$ in the direction of $$\vec{v}$$**
- ✓ **+ 2 pts $$D_{\vec{v}}f(0,-1) = \nabla f_{(0,-1)} \cdot \vec{u}$$**
- ✓ **+ 1 pts $$D_{\vec{v}}f(0,-1) = \frac{-57}{5}$$, or correct given previous computations**
- **- 1 pts** Small algebra mistake
- **- 2 pts** Two small algebra mistakes
- **+ 3 pts** (partial) Dotted the gradient with $$\vec{v}$$ instead of with $$\vec{u}$$

4. *(25 points) Consider the function*

$$f(x,y) = x^3 - 3x - 2y^3 + 3y^2 - 1$$

(a) *Find*

$$\frac{\partial f}{\partial x} \quad \& \quad \frac{\partial f}{\partial y}.$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 3 \\ \frac{\partial f}{\partial y} &= -6y^2 + 6y \end{aligned}}$$

(b) *Find $\vec{\nabla} f$ at the point $(0,-1)$.*

$\vec{\nabla} f$ at the point $(0,-1) = \langle \frac{\partial f}{\partial x}(0,-1), \frac{\partial f}{\partial y}(0,-1) \rangle = \boxed{\langle -3, -12 \rangle}$

(c) *If $\vec{v} = \langle 3, 4 \rangle$, then find the directional derivative of f in the direction of $\vec{v}$ at the point $(0,-1)$.*

find unit vector: $\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$, so $\vec{v} = \vec{e}_{\vec{v}} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$$D_{\vec{v}} f(0,-1) = \underset{\vec{\nabla} f @(0,-1)}{\langle -3, -12 \rangle} \cdot \underset{\text{unit vector in direction of } \vec{v}}{\langle \frac{3}{5}, \frac{4}{5} \rangle} = -\frac{9}{5} + \frac{-48}{5} = \boxed{\frac{-57}{5}}$$

(d) *At the point $(0,-1)$, what is a direction you could begin to walk in if you want to walk along an isocline (i.e. leaving the height unchanged)?*

$\vec{\nabla} f_{(0,-1)} = \langle -3, -12 \rangle$ points in the direction of maximum rate of increase, so we need normal vector.

$\langle -3, -12 \rangle \cdot \langle -4, 1 \rangle = 12 - 12 = 0$, $\sqrt{4^2+1^2} = \sqrt{17} = \boxed{\langle \frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle}$

(e) *Are there any directions $\vec{u}$ where the directional derivative $D_{\vec{u}} f(0,-1)$ is greater than 13?*

$D_{\vec{v}} f(0,-1) = \langle -3, -12 \rangle \cdot \vec{v} > 13$

No, because the greatest number we can achieve is 12, where $\vec{v} = \langle 0, -1 \rangle$

4.4 Isocline 5 / 5

- ✓ **+ 3 pts** **Must walk perpendicular to the gradient**
- ✓ **+ 1 pts** **Direction vector will satisfy $$\langle v_1, v_2 \rangle \cdot \langle -3, -12\rangle = 0$$**
- ✓ **+ 1 pts** **Any nonzero vector of the form $$\langle -4c, c\rangle$$ works.**
- **+ 0 pts** Blank or incorrect

4\. *(25 points) Consider the function*

$$f(x,y) = x^3 - 3x - 2y^3 + 3y^2 - 1$$

(a) Find

$$\frac{\partial f}{\partial x} \quad \& \quad \frac{\partial f}{\partial y}.$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 3 \\ \frac{\partial f}{\partial y} &= -6y^2 + 6y \end{aligned}}$$

(b) Find $\vec{\nabla} f$ at the point $(0,-1)$.

$\vec{\nabla} f$ at the point $(0,-1) = \left\langle \frac{\partial f}{\partial x}(0,-1), \frac{\partial f}{\partial y}(0,-1) \right\rangle = \boxed{\langle -3, -12 \rangle}$

(c) If $\vec{v} = \langle 3,4 \rangle$, then find the directional derivative of f in the direction of $\vec{v}$ at the point $(0,-1)$.

find unit vector: $\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$, so $\vec{v} = \vec{e}_{\vec{v}} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\vec{v}} f(0,-1) = \langle -3,-12 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = -\frac{9}{5} + \frac{-48}{5} = \boxed{\frac{-57}{5}}$

$\vec{\nabla} f_{@(0,-1)}$ unit vector in direction of $\vec{v}$

(d) At the point $(0,-1)$, what is a direction you could begin to walk in if you want to walk along an isocline (i.e. leaving the height unchanged)?

$\vec{\nabla} f_{(0,-1)} = \langle -3, -12 \rangle$ points in the direction of maximum rate of increase, so we need normal vector.

$\langle -3,-12 \rangle \cdot \langle -4, 1 \rangle = 12 - 12 = 0$, $\sqrt{4^2+1^2} = \sqrt{17} = \boxed{\langle \frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle}$

(e) Are there any directions $\vec{u}$ where the directional derivative $D_{\vec{u}} f(0,-1)$ is greater than 13?

$D_{\vec{v}} f(0,-1) = \langle -3, -12 \rangle \cdot \vec{v} > 13$

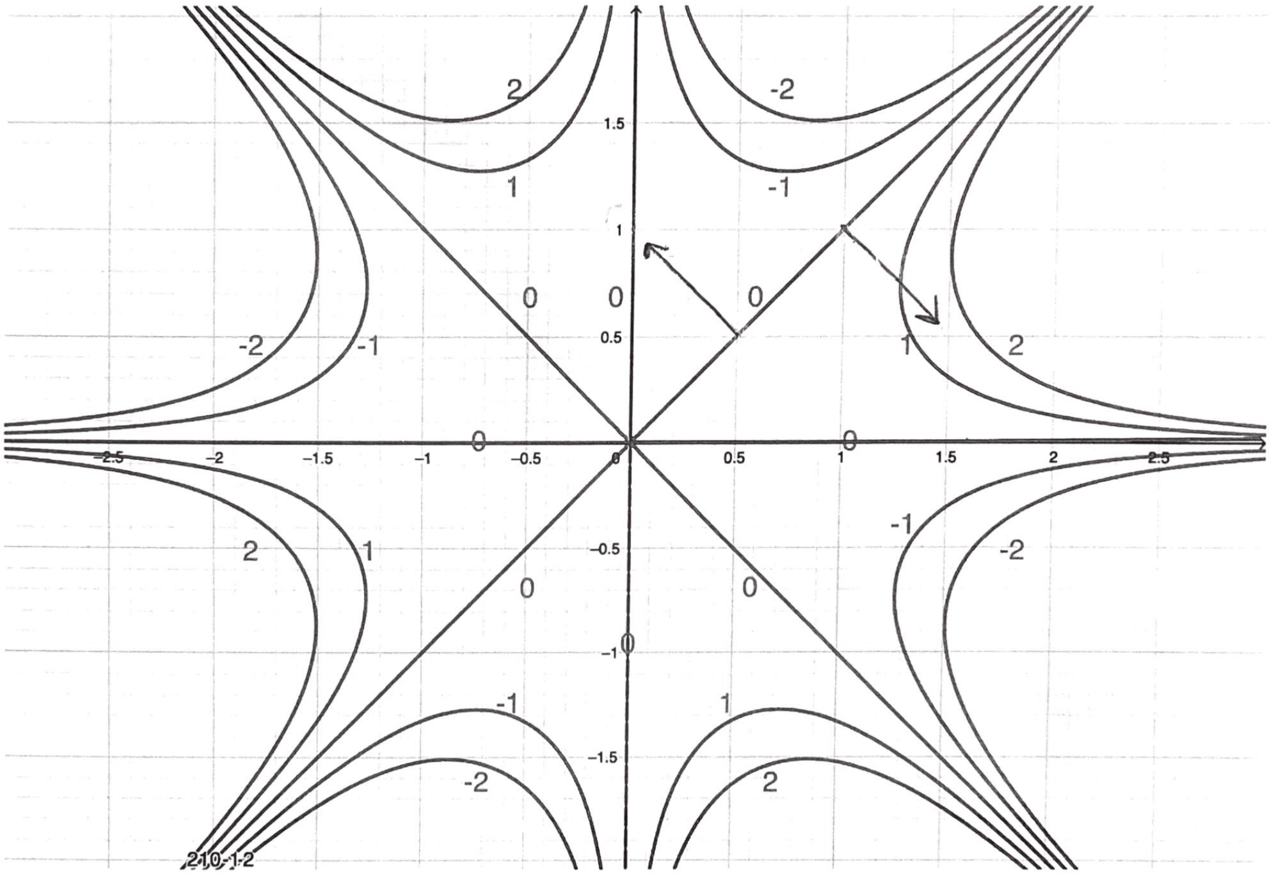
No, because the greatest number we can achieve is 12, where $\vec{v} = \langle 0,-1 \rangle$

4.5 Max slope **2 / 5**

- **+ 2 pts** Maximum value of the direction derivative is the magnitude of the gradient
- **+ 2 pts** At $$(0,-1)$$, $$||\vec{\nabla{f}}|| = \sqrt{153}$$
- **+ 1 pts** $$\sqrt{153} < 13$$, so the directional derivative cannot be greater than $$13$$
- **- 1 pts** Small algebra mistake
- **+ 0 pts** Blank or incorrect
- ✓ **+ 2 pts (partial) Correct answer but no justification or incorrect justification.**

The direction derivative can be greater than 12.

5. *(25 points) Consider the contour plot of a function $f(x,y)$ given in the picture.*



(a) *(5 Points each) Draw vectors on the picture indicating the gradients at $(\frac{1}{2},\frac{1}{2})$ and at $(1,1)$. Clearly indicate the directions and make sure your pictures indicate any differences in length.*

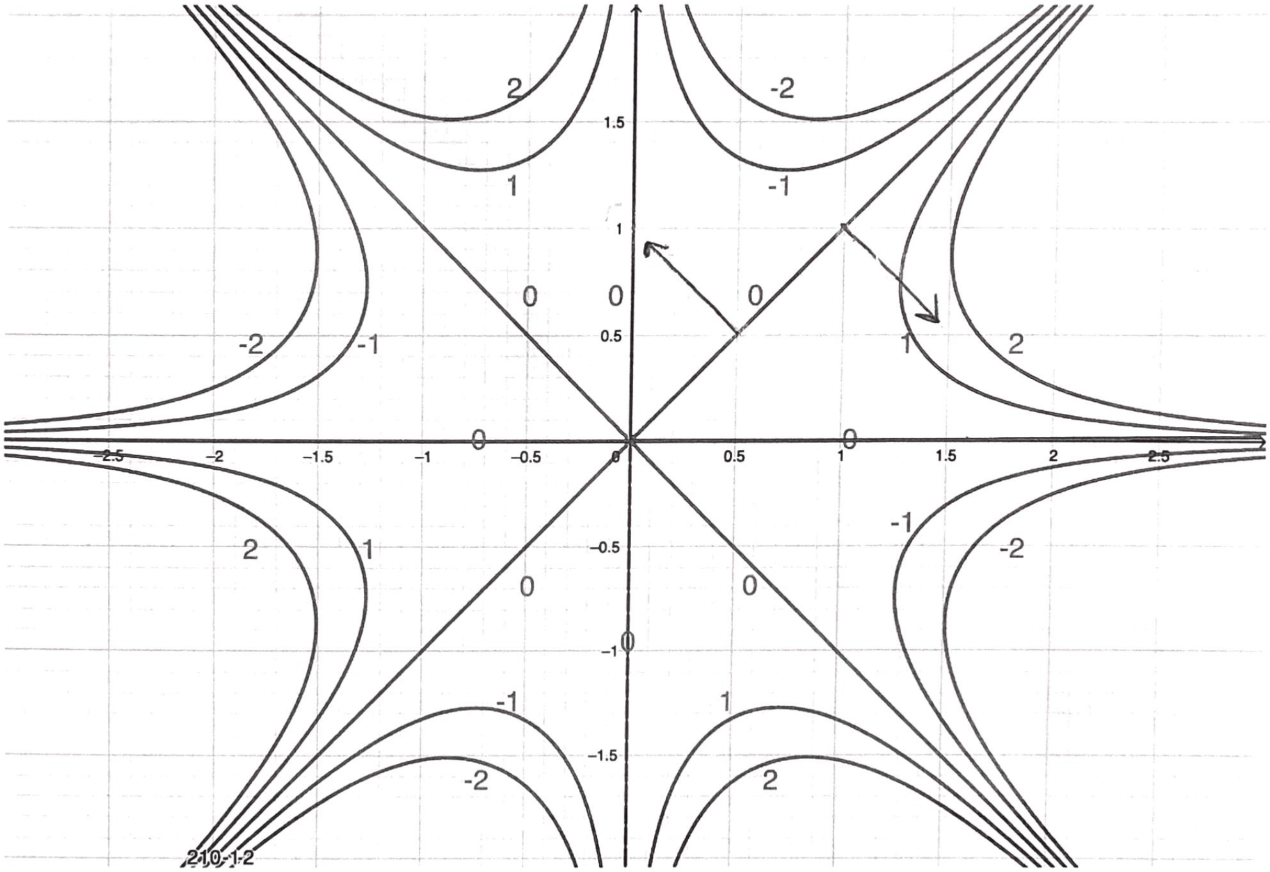
(b) *(5 Points) How did you decide on the directions and relative lengths?*

I decided on the directions by making the vectors orthogonal to the level curves at their respective points and pointing in the direction of steepest ascent. So, for (0.5, 0.5) this was in the northwest direction towards the 2, and for (1, 1) in the southeast direction towards the 2. Their respective lengths are relatively the same, since they both lie on the same curve, which will have the same maximum slope of a tangent line, which is equal to the length of the gradient.

5.1 Gradients 6 / 10

- + **10 pts** Correct
- + **2 pts** One perpendicular
- ✓ + **4 pts** **Both perpendicular**
- ✓ + **2 pts** **One points in right direction**
- + **4 pts** Both point in right direction
- + **2 pts** Lengths scaled
- - **1 pts** Drawing mistake
- + **0 pts** Incorrect or unclear

5. *(25 points) Consider the contour plot of a function $f(x, y)$ given in the picture.*



(a) *(5 Points each) Draw vectors on the picture indicating the gradients at $(\frac{1}{2}, \frac{1}{2})$ and at $(1, 1)$. Clearly indicate the directions and make sure your pictures indicate any differences in length.*

(b) *(5 Points) How did you decide on the directions and relative lengths?*

I decided on the directions by making the vectors orthogonal to the level curves at their respective points and pointing in the direction of steepest ascent. So, for (0.5, 0.5) this was in the northwest direction towards the 2, and for (1, 1) in the southeast direction towards the 2. Their respective lengths are relatively the same, since they both lie on the same curve, which will have the same maximum slope of a tangent line, which is equal to the length of the gradient.

5.2 Reasoning 4 / 5

✓ + **5 pts** **Correct**

✓ - **1 pts** **Length argument wrong**

+ **2 pts** Recognized role of steepest ascent

+ **1 pts** Only addressed length

+ **0 pts** Not correct

+ **2 pts** Recognizes perpendicular to level curve

- **2 pts** Wrong wording for directions *(c) (5 points each) If this were the topographic map of a mountain range, would you describe the geography near $(0, 0)$ as fairly flat or steep?*

If this were a topographic map of a mountain range, I would describe the geography near $(0,0)$ as fairly flat because there are not many different level curves around the point $(0,0)$, meaning that the slope of the mountain wouldn't have much change around the point. The closest level curves are pretty far away, so the area around $(0,0)$ must be fairly flat.

What about near $(\frac{3}{2}, \frac{3}{2})$?

The geography near $(\frac{3}{2}, \frac{3}{2})$ would be best described as fairly steep because there are many level curves within close proximity of the point, meaning that the slope of the mountain would be rapidly changing around the point $(\frac{3}{2}, \frac{3}{2})$. Therefore, since the level curves are all very close, the area around $(\frac{3}{2}, \frac{3}{2})$ must be fairly steep.

5.3 Geometry 5 / 5

✓ **+ 5 pts** **Correct**

+ 0 pts Wrong geometry

(c) *(5 points each) If this were the topographic map of a mountain range, would you describe the geography near* $(0, 0)$ *as fairly flat or steep?*

If this were a topographic map of a mountain range, I would describe the geography near $(0,0)$ as fairly flat because there are not many different level curves around the point $(0,0)$, meaning that the slope of the mountain wouldn't have much change around the point. The closest level curves are pretty far away, so the area around $(0,0)$ must be fairly flat.

What about near $(\frac{3}{2}, \frac{3}{2})$*?*

The geography near $(\frac{3}{2}, \frac{3}{2})$ would be best described as fairly steep because there are many level curves within close proximity of the point, meaning that the slope of the mountain would be rapidly changing around the point $(\frac{3}{2}, \frac{3}{2})$. Therefore, since the level curves are all very close, the area around $(\frac{3}{2}, \frac{3}{2})$ must be fairly steep.

5.4 Near 1.5, 1.5 5 / 5

- ✓ **+ 5 pts** Correct
- **+ 0 pts** Wrong geometry