MATH 32A FINAL EXAM JUNE 9, 2020

NAME: CHRISTIAN B AGUILAR ID: 405-302-745

SIGNATURE: Christian B Aguilar

To get credit for a problem, you must show all of your reasoning and calculations. You may consult your books, notes, calculator, any materials from the CCLE site, the professor, or your TA. You may not collaborate or ask questions online. $\boxed{\textit{Box your final answer.}}$

If you cannot find a vector that you need for a later part of a problem, you may use the vector $\langle 1, 2, 3\rangle$.

If you cannot find a point that you need for a later part of a problem, you may use the point $(1, 1, 1)$.

Circle your section:

Section:	**Tuesday:**	**Thursday:**	**TA:**
	2A	2B	Alexander Johnson
	2C	2D	Francis White
	2E	2F	Jason Snyder

1. Consider the function $f(x, y) = x^2 + y^2$. Find the extrema of f subject to $3x + 4y = 5$ in two ways:

(a) (10 points) Parameterize the constraint line & plug into f.

$t = x$

$y = \frac{5}{4} - \frac{3}{4}t$

$f(t) = t^2 + (\frac{5}{4} - \frac{3}{4}t)^2$ $\quad r(t) = \langle t, \frac{5}{4} - \frac{3}{4}t \rangle$

$\frac{df}{dt} = 2t + 2(\frac{5}{4} - \frac{3}{4}t)(-\frac{3}{4})$

$= 2t - \frac{3}{2}(\frac{5}{4} - \frac{3}{4}t)$

$= 2t - \frac{15}{8} + \frac{9}{8}t$

$= \frac{16}{8}t - \frac{15}{8} + \frac{9}{8}t$

$= \frac{25t}{\cancel{8}} - \frac{15}{8} \rightarrow \frac{25t}{8} = \frac{15}{8}$

$t = \frac{15}{25} = \boxed{x = \frac{3}{5}}$

$y = 5$

$) + 4y = 5$

$+ 4y = 5$

$4y = \frac{25}{5} - \frac{4}{5}$

$4y = \frac{16}{5} \rightarrow \boxed{y = \frac{4}{5}}$

$f(x, y) = (\frac{3}{5})^2 + (\frac{4}{5})^2$

$= \frac{4}{25} + \frac{16}{25} = \frac{25}{25} = \boxed{1} = f(x, y)$

$\boxed{\text{Extrema: } f(3/5, 4/5) = 1}$

$\boxed{(3/5, 4/5)}$

(b) (10 points) Lagrange Multipliers

$f(x,y) = x^2 + y^2$ $\quad g(x,y) = 3x + 4y - 5 = 0$

$\nabla f = \langle 2x, 2y \rangle$ $\quad \nabla g = \langle 3, 4 \rangle$

$\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y \rangle = \lambda \langle 3, 4 \rangle$

$2x = \lambda 3 \quad 2y = \lambda 4$

$\frac{2x}{3} = \lambda = \frac{2y}{4}$

$\rightarrow 4x = 3y$

$\boxed{x = \frac{3y}{4}}$

$3x + 4y = 5$

$3\left(\frac{3y}{4}\right) + 4y = 5$

$\frac{9y}{4} + 4y = 5$

$\frac{25y}{4} = \frac{20}{4}$

$y = \frac{20}{25} = \frac{4}{5}$

$\boxed{y = \frac{4}{5}}$

$x = \frac{3}{4} \cdot \left(\frac{4}{5}\right)$

$\boxed{x = \frac{3}{5}}$

$\boxed{\text{Extrema @ } \left(\frac{3}{5}, \frac{4}{5}\right),\ f\left(\frac{3}{5}, \frac{4}{5}\right) = 1}$

$f(x,y) = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$

$\frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$

(c) (5 points) Is the point you found a maximum or a minimum? Explain how you know, and explain why you weren't guaranteed to have a global maximum and minimum.

• Checking the 2nd derivative of the parametrization

$$f(t) = t^2 + \left(\frac{5}{4} - \frac{3}{4}t\right)^2$$

$$\frac{df}{dt} = 2t + 2\left(\frac{5}{4} - \frac{3}{4}t\right)\left(\frac{-3}{4}\right)$$

$$\frac{d^2f}{dt} \quad 2t + \left(\frac{10}{4} - \frac{6t}{4}\right)\left(\frac{-3}{4}\right)$$

$$\frac{df^2}{dt} \quad 2t \quad \frac{-30}{16} + \frac{18t}{16}$$

$$2 + \frac{18}{16}$$

$$2 + \frac{9}{8} \rightarrow \frac{16}{8} + \frac{9}{8} = \frac{25}{8}$$

2nd derivative of parametrization is **positive**, therefore the point is a local minimum on the constraint.

We aren't guaranteed to have a global maximum because the constraint is unbounded (and can go out forever in both directions).

We aren't guaranteed the global minimum of $f(x,y)$ because our constraint might not contain that point.

2. Let

$$f(x, y) = xe^{xy}.$$

(a) (10 points) Find the gradient of f.

$f_x = (1)e^{xy} + xe^{xy}y$
$= e^{xy} + xye^{xy}$

$f_y = xe^{xy}x$
$= x^2e^{xy}$

gradient: $\langle \frac{df}{dx}, \frac{df}{dy} \rangle$

$\boxed{= \langle e^{xy} + xye^{xy}, x^2e^{xy} \rangle}$

(b) (10 points) Find the linear approximation to $f(x, y)$ at the point $(1, 0)$.

Tangent plane @ $(a,b) = \frac{df}{dx}(a,b)(x-a) + \frac{df}{dy}(a,b)(y-b) + f(a,b)$

$\therefore f(a+\Delta x, b+\Delta y) \approx f_x(a,b)(\Delta x) + f_y(a,b)\Delta y + f(a,b)$

$\approx (e^{ab} + abe^{ab})(\Delta x) + a^2e^{ab}(\Delta y) + ae^{ab}$

$\approx (e^{(1)(0)} + (1)(0)e^{(1)(0)})\Delta x + (1)^2e^{(1,0)}\Delta y + (1)e^{(1,0)}$

$\boxed{f(x+\Delta x, y+\Delta y) \approx (1)\Delta x + (1)\Delta y + 1}$

(c) (5 points) Use the linear approximation to estimate $f(1.1, -.1)$.

$x = 1.1$ $\quad \Delta x = .1$

$a = 1$

$y = -.1$ $\quad \Delta y = -.1$

$z \approx .1 + -.1 + 1$

$\boxed{z \approx 1}$

3. (2 points each) True/False! Circle the appropriate answer.

No justification is needed here.

(1) For any two vectors $\vec{v}$ and $\vec{u}$, $\vec{v}\cdot\vec{u}=\vec{u}\cdot\vec{v}$.	True	False
(2) If $f(x,y)$ and $g(x,y)$ are continuous at (a,b), then the function $f(x,y)\cdot g(x,y)$ is continuous at (a,b)	True	False
(3) For any two vectors $\vec{u}$ and $\vec{v}$, $$\vec{u}\times\vec{v}=\lVert\vec{u}\rVert\cdot\lVert\vec{v}\rVert\cdot\sin\theta,$$ where θ is the angle between $\vec{u}$ and $\vec{v}$.	True	False
(4) The x-component of a vector $\vec{v}$ always equals the dot product $\vec{v}\cdot\vec{\imath}$.	True	False
(5) A continuous function on a closed but **not** bounded region in $\mathbb{R}^2$ cannot have an absolute maximum and minimum.	True	False
(6) The gradient of f is tangent to the level curves.	True	False
(7) The curvature of a curve in space can be negative.	True	False
(8) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three vectors all of whose components are integers. Then $\vec{u}\cdot(\vec{v}\times\vec{w})$ is an integer.	True	False
(9) For any two vectors $\vec{v}$ and $\vec{u}$, $\vec{v}\times\vec{u}=-\vec{u}\times\vec{v}$.	True	False
(10) If the limits of $f(x,y)$ along all lines through a point (a,b) exist and agree, then the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.	True	False
(11) The cross product of two unit vectors that are not parallel is always unit vector.	True	False
(12) The set of points $\{(x,y)\mid 0<x^2+y^2-9\le 16\}$ is closed.	True	False
(13) A continuous function on a closed and bounded region has an absolute maximum and minimum.	True	False
(14) If a level curve intersects itself at a point so that there are two distinct tangent directions, then the point is a critical point	True	False
(15) Given a vector $\vec{v}$ and a non-zero vector $\vec{w}$, we always have $\lVert\vec{v}_{\perp\vec{w}}\rVert\le\lVert\vec{v}\rVert$.	True	False

$\langle v_x, v_y$ $\quad \langle u_x, u_y\rangle \times \langle v_x, v_y\rangle = \begin{vmatrix} i & j & k \\ u_x & u_y & 0 \\ v_x & v_y & 0 \end{vmatrix}$ $\quad \langle 0, 0, u_xv_y - u_yv_x\rangle$

4. Consider the hyperboloid of one sheet described by the equation

$$x^2 + y^2 - z^2 = 4.$$

(a) (10 points) The equation defines z implicitly in terms of x and y. Find

$\boxed{\frac{\partial z}{\partial x} = \frac{x}{z}}$

$\frac{d}{dx}\left(x^2+y^2-z^2\right) = \frac{d}{dx}(4)$

$2x + (0) - 2z\frac{dz}{dx} = 0$

$-2z\frac{\partial z}{\partial x} = -2x$

$\frac{\partial z}{\partial x} = \frac{2x}{-2z} \rightarrow \frac{x}{z}$

$\boxed{\frac{\partial z}{\partial y} = \frac{y}{z}}$

$2y - 2z\frac{dz}{dy} = 0$

$2y = 2z\frac{dz}{dy}$

$\frac{y}{z} = \frac{dz}{dy}$

(b) (10 points) Find the equation of the tangent plane at the point $(2, -2, -2)$.

a b c

$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + c$

$\frac{x}{z} = \frac{2}{-2} = -1 = (-1)(x-2) + (1)(y-(-2)) + -2$

$\frac{y}{z} = \frac{-2}{-2} = 1$

$\rightarrow -(x-2) + (y+2) - 2$

$\boxed{z = -(x-2) + (y+2) - 2}$

5. Consider the function
$$f(x, y) = x^4 - 2x^2 + y^4 - 8y^2.$$

(a) (20 points) Find the 9 critical points.

· Find 9 crit points (where $f_x = 0$ and $f_y = 0$)

$f_x = 4x^3 - 4x$ $\quad$ $f_y = 4y^3 - 16y$

$f_x = 4x(x^2-1)$ $\quad$ $f_y = 4y(y^2-4)$

$f_x = 4x(x+1)(x-1)$ $\quad$ $f_y = 4y(y+2)(y-2)$

crit points:

$x = 0$ $\quad$ $y = 0$

$x = 1$ $\quad$ $y = 2$

$x = -1$ $\quad$ $y = -2$

Both f_x and $f_y = 0$ at ...

(0,0)	(1,0)	(-1,0)
(0,2)	(1,2)	(-1,2)
(0,-2)	(1,-2)	(-1,-2)

(0,0,0)

(b) (7 points) For this function, what is the discriminant D that plays a role in the 2nd derivative test?

Discriminant $D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 12x - 4$

$f_{yy} = 12y - 16$

$f_{xy} = 0$

$$\boxed{D = (12x-4)(12y-16)}$$

(c) (3 points each) Find one local maximum, one local minimum, and one saddle point for this function (you do not need to classify all of the critical points).

Test $(0,0)$

$D(0,0) = (-4)(-16) = +64$ Positive →

$f_{xx}(0,0) = -4$ negative →

$\boxed{(0,0) \text{ is a local maximum}}$

Test $(0,2)$: $D(0,2) = (-4)(24-16)$

$= (-4)(8) = -32$ negative, ∴

$\boxed{(0,2) \text{ is a saddle point}}$

Test $(1,2)$: $D(1,2) = (12-4)(8)$

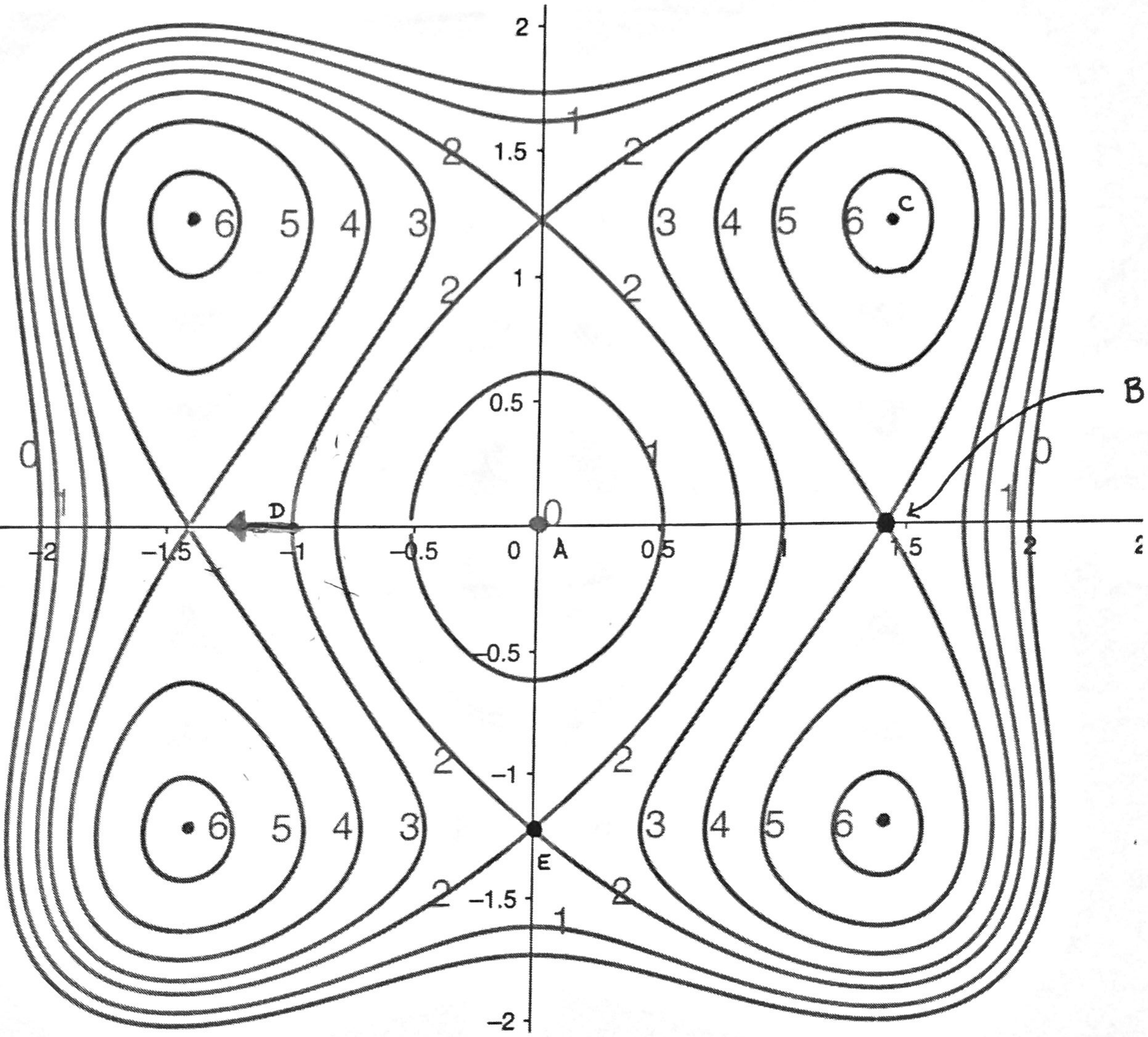
$(8)(8)$ → positive

f_{xx} is positive

∴

$\boxed{(1,2) \text{ is a local minimum}}$

6. Consider the contour plot of a function $f(x, y)$ below (where the contour labels are the larger numbers written in red next to the curves):



(a) (3 points each) The points labeled A, B, and C are critical points (and the only critical points within the next closest contour). Classify them as maxima, minima, or saddles. **Briefly** say how you know.

A is local minimum. It is a point with 0 vector gradient and lower elevation than its surroundings

B: Is a saddle. 2 level curves intersect here, giving 2 distinct tangent lines

C: Local max. Point w/ $\vec{0}$ gradient. Higher elevation than it's surroundings level curves.

(b) (3 points) Draw the gradient at the point labeled D. No justification is needed.

(c) (3 points) What is the gradient at the point labeled E? **Briefly** explain how you know.

c:) Gradient at point E is zero.
It is a saddle point. In
the immediate vicinity of E,
There is no slope.
$\frac{\partial f}{\partial x}(E)$ and $\frac{\partial f}{\partial y}(E) = 0$

7. Consider the curve given by

$$\vec{r}(s) = \left\langle \frac{4s}{5}, 3\sin\left(\frac{s}{5}\right), -3\cos\left(\frac{s}{5}\right)\right\rangle.$$

(a) (5 points) Show that s is the arc-length parameter.

. Arc length parameterization has a constant speed of 1.

$\frac{d\vec{r}}{ds} = \left\langle \frac{4}{5}, \frac{3}{5}\cos\left(\frac{s}{5}\right), +\frac{3}{5}\sin\frac{s}{5}\right\rangle$ $\langle \frac{4}{5}, \frac{3}{5}\cos, \frac{3}{5}\sin\rangle$

$\langle 0, -\sin, +\cos$

speed = magnitude of velocity

$= \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\cos\frac{s}{5}\right)^2 + \left(\frac{+3}{5}\sin\frac{s}{5}\right)^2}$

$= \sqrt{\frac{16}{25} + \left(\frac{9}{25}\cos^2\frac{s}{5}\right) + \left(\frac{9}{25}\sin^2\frac{s}{5}\right)}$

$= \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}}$ $\boxed{= 1}$ → is an arc-length parameterization.

(b) (10 points) Find the unit normal and curvature as a function of s.

. Since magnitude of velocity is 1, then the velocity is also the unit tangent vector.

$N(t) = \frac{T'(t)}{\|T'(t)\|}$ $T = \left\langle \frac{4}{5}, \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\sin\frac{s}{5}\right\rangle$

$T'(s) = \left\langle 0, \frac{3}{25}\left(-\sin\left(\frac{s}{5}\right)\right), \frac{3}{25}\cos\frac{s}{5}\right\rangle$

$\boxed{N(s) = \left\langle 0, -\sin\frac{s}{5}, \cos\frac{s}{5}\right\rangle}$

$k(s) = \left\|\frac{dT}{ds}\right\| = \sqrt{\left(\frac{3}{25}\right)^2\left(\sin^2\frac{s}{5}\right) + \frac{3}{25}\cos^2\frac{s}{5}}$

$= \sqrt{\left(\frac{3}{25}\right)^2 +}$

$\boxed{k(s) = \frac{3}{25}}$

8. Consider the function

$$f(x, y) = \frac{x^2}{x^2 + y^2}$$

(a) (5 points) Where is f continuous and why?

x^2: continuous everywhere (polynomial)

x^2+y^2: continuous everywhere (addition of polynomials)

$\frac{x^2}{x^2+y^2}$: continuous where x^2+y^2 does not vanish (division of polynomials)

$\boxed{\therefore \text{ Continuous everywhere where } x^2+y^2 \text{ does not vanish}}$

(b) (10 points) Is there a value a such that the function

$$\tilde{f}(x, y) = \begin{cases} \frac{x^2}{x^2+y^2} & (x, y) \neq (0, 0) \\ a & (x, y) = (0, 0) \end{cases}$$

is continuous? If yes, find it and show continuity. If no, show why not.

Def. Continuity: $f(a,b)$ exists

$\lim_{f(x,y)\to(a,b)} f(x,y)$ exists ← we'll be testing this.

$\lim_{f(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Test for all lines going through $(0,0)$ $y=mx$: $\rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+(mx)^2} \rightarrow \frac{x^2}{x^2+x^2m^2}$

$\rightarrow \frac{x^2}{x^2(1+m^2)} \rightarrow \lim_{(x,y)\to(0,0)} \rightarrow \frac{1}{(1+m^2)}$

$\boxed{\text{Limit depends on } m, \text{ does not exist, } \therefore \text{ no value of } a \text{ can satisfy continuity}}$

9. A particle moves along the curve $y = x^3 - 4x$, shown below. Using x as the parameter, find

(a) (5 points) The velocity of the particle

$r(x)$: $x = x$, $y = x^3 - 4x$

$r(x) = \langle x, x^3 - 4x \rangle$

$\frac{dr}{dx}(x) = \langle 1, 3x^2 - 4 \rangle$

$$\boxed{\vec{v}(x) = \langle 1, 3x^2 - 4 \rangle}$$

(b) (5 points) the acceleration of the particle

$a(t) = r''(t) = \vec{v}'(t)$

$$\boxed{\vec{a}(x) = \langle 0, 6x \rangle}$$

$$\begin{array}{ccc} i & j & k \\ 1 & 3x^2-4 & 0 \\ 0 & 6x & 0 \end{array}$$

$k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

$\frac{\|r'(t) \times r''(t)\|}{\left(\sqrt{1 + (3x^2-4)^2}\right)^3} = \frac{16H}{(\sqrt{9x^4 - 24}}$

(c) (5 points) the curvature of the curve as a function of x.

Curvature for point $(x, f(x))$ on graph of $y = f(x)$ =

$k(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$

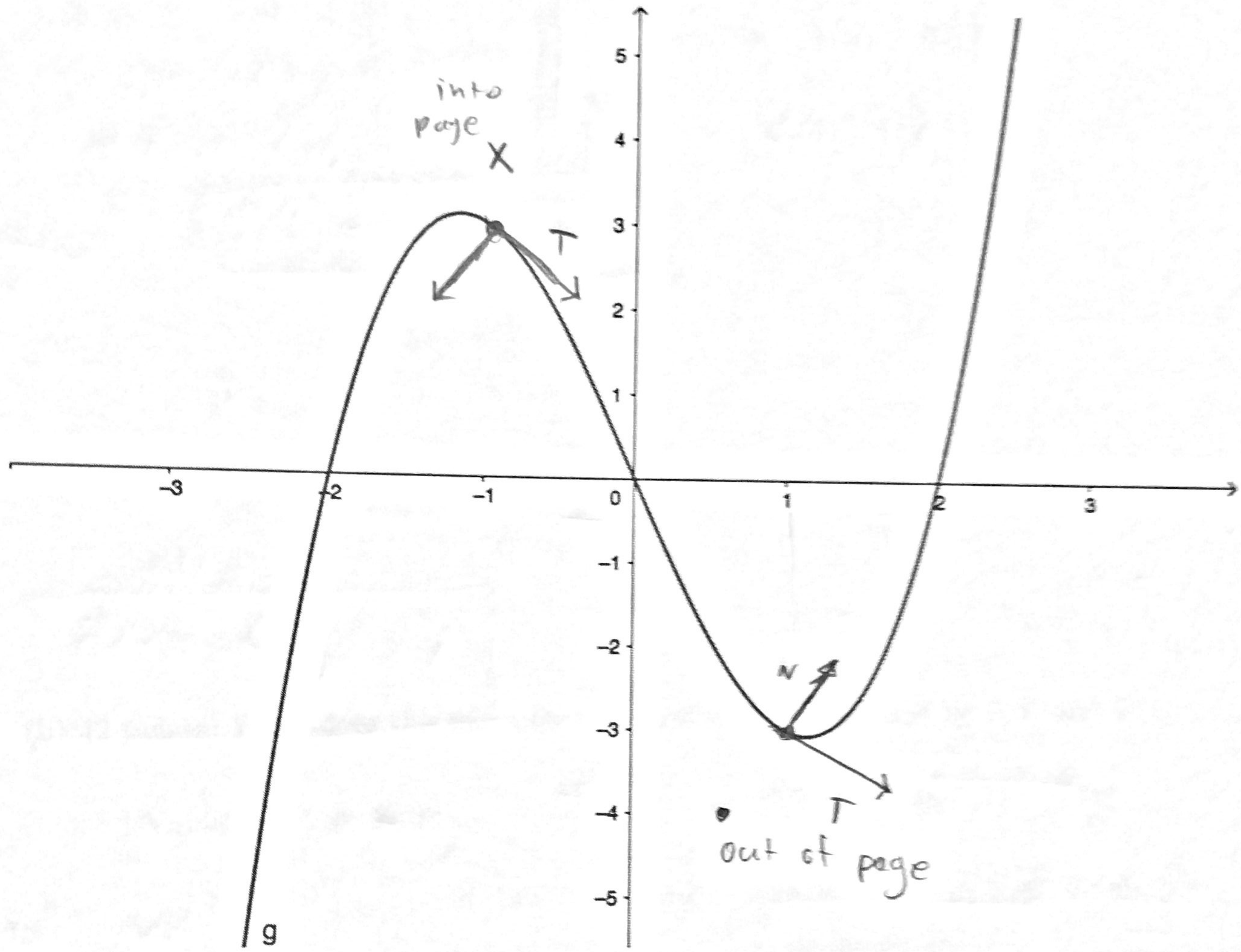
$f'(x) = 3x^2 - 4$

$f''(x) = 6x$

$(3x^2 - 4)^2 = (9x^4 - 24x^2 + 16)$

$$\boxed{k(x) = \frac{|6x|}{(9x^4 - 24x^2 + 17)^{3/2}}}$$

(d) (5 points each) On the picture below, draw the unit tangent and unit normal vectors at $(-1, 3)$ and $(1, -3)$. Next to each point, indicate if the binormal vector goes up out of the page or down into the page.



10. Let $\vec{u} = \langle 3, -2, a\rangle$, $\vec{v} = \langle 1, 0, 1\rangle$, and $\vec{w} = \langle 0, 1, 1\rangle$.

(a) (8 points) For what a is the vector triple product $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$?

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \quad i(-1) - j(1) + k(1)$$

$$\vec{v} \times \vec{w} = \langle -1, -1, 1\rangle$$

$$\langle 3, -2, a\rangle \cdot \langle -1, -1, 1\rangle$$

$$= 3(-1) + (-2)(-1) + a = 0$$

$$-3 + 2 + a = 0$$

$$-1 + a = 0$$

$$\boxed{a = 1}$$

(b) (2 points) What does this mean for the parallelepiped spanned by $\vec{u}$, $\vec{v}$, and $\vec{w}$?

when $a = 0$, $\vec{u}$ is orthogonal to the normal vector of the plane formed by $\vec{v}$ and $\vec{w}$. As a result, $\vec{u}$ sits on the same plane. The triple product, representing the $\boxed{\text{volume}}$ of the parallelepiped, will be $\boxed{\text{zero.}}$