1. For all sub-parts of this question, let $\mathbf{u} = \langle 3, -9, 5\rangle$ and $\mathbf{v} = \langle 3, 0, 4\rangle$.

(a) (5 points) Compute $\mathbf{u} - 2\,\mathbf{v}$

$$\langle 3, -9, 5\rangle - 2\langle 3, 0, 4\rangle$$

$$\langle 3, -9, 5\rangle - \langle 6, 0, 8\rangle$$

$$= \boxed{\langle -3, -9, -3\rangle}$$

(b) (5 points) Write $\mathbf{u}$ as a sum: $\mathbf{u} = \mathbf{u}_{\parallel \mathbf{v}} + \mathbf{u}_{\perp \mathbf{v}}$ where $\mathbf{u}_{\parallel \mathbf{v}}$ is parallel to $\mathbf{v}$ and $\mathbf{u}_{\perp \mathbf{v}}$ is perpendicular to $\mathbf{v}$.

$$u_{\parallel v} = (u \cdot e_v)e_v \qquad u_{\perp v} = u - u_{\parallel v}$$

$$e_v = \frac{1}{\sqrt{9+0+16}} \langle 3, 0, 4\rangle$$

$$= \left\langle \frac{3}{\sqrt{25}}, 0, \frac{4}{\sqrt{25}} \right\rangle$$

$$u_{\parallel v} = \left(\langle 3,-9,5\rangle \cdot \left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle\right)\left(\left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle\right)$$

$$u_{\parallel v} = \left(\frac{9}{5} + 0 + 4\right)\left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle$$

$$u = \langle 3, -9, 5\rangle$$

$$u_{\parallel v} = \left(\frac{29}{5}\right)\left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle = \left\langle \frac{87}{25}, 0, \frac{116}{25}\right\rangle$$

$$u_{\perp v} = u - u_{\parallel v}$$

$$u_{\perp v} = \langle 3, -9, 5\rangle - \left\langle \frac{87}{25}, 0, \frac{116}{25}\right\rangle$$

$$\boxed{u = \left\langle \frac{87}{25}, 0, \frac{116}{25}\right\rangle + \left\langle -\frac{12}{25}, -9, \frac{9}{25}\right\rangle}$$

(c) (5 points) Let $P = (5, 11, -3)$ and $Q = (8, 11, 1)$. Is the vector $\overrightarrow{PQ}$ equivalent to either $\mathbf{u}$ or $\mathbf{v}$? Justify your answer.

$$\overrightarrow{PQ} = \langle 8-5, 11-11, 1-(-3)\rangle$$

$$\boxed{\overrightarrow{PQ} = \langle 3, 0, 4\rangle}$$

$$e_{\overrightarrow{PQ}} = \frac{1}{\sqrt{9+16}}\langle 3, 0, 4\rangle$$

$$e_{\overrightarrow{PQ}} = \left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle$$

The vector $\overrightarrow{PQ}$ is equivalent to v because their components are the same, as well as their unit vectors.

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2. (10 points) Find the equation of the plane passing through the points $P = (1, 2, 1)$, $Q = (2, 2, 4)$ and $R = (-1, 2, 3)$

$\vec{PQ} = \langle 2-1, 2-2, 4-1 \rangle = \langle 1, 0, 3 \rangle$

$\vec{QR} = \langle -1-2, 2-2, 3-4 \rangle = \langle -3, 0, -1 \rangle$

$\vec{PQ}$ and $\vec{QR}$ are 2 vectors in the plane.

$$\underbrace{\vec{PQ} \times \vec{QR}}_{\text{cross gives us a normal vector}} = \begin{vmatrix} i & j & k \\ 1 & 0 & 3 \\ -3 & 0 & -1 \end{vmatrix} = i \begin{vmatrix} 0 & 3 \\ 0 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix}$$

$$= (0-0)i - (-1+9)j + (0-0)k$$

$$= -8j \rightarrow \langle 0, -8, 0 \rangle = \vec{n}$$

Check:

$\langle 0, -8, 0 \rangle \cdot \langle 1, 0, 3 \rangle = 0 + 0 + 0 = 0$

$\langle 0, -8, 0 \rangle \cdot \langle -3, 0, -1 \rangle = 0 + 0 + 0 = 0$

$\vec{n} = \langle a, b, c \rangle$ point $(x_0, y_0, z_0) \rightarrow P(1, 2, 1)$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

equation of the plane

$0(x-1) - 8(y-2) + 0(z-1) = 0$

$-8y + 16 = 0$

$-8y = -16$

$\boxed{y = 2}$

3. (10 points) Find a parametrization of the tangent line to the curve given by $\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$ at the point $t = \pi/2$

$r(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$

$r(\pi/2) = \langle 3\cos(\frac{\pi}{2}), 5\sin(\frac{\pi}{2}), 4\cos(\frac{\pi}{2})\rangle$

$r(\pi/2) = \langle 0, 5, 0\rangle$

$r'(t) = \langle -3\sin(t), 5\cos(t), -4\sin(t)\rangle$

$r'(\pi/2) = \langle -3\sin(\pi/2), 5\cos(\pi/2), -4\sin(\pi/2)\rangle$

$r'(\pi/2) = \langle -3, 0, -4\rangle$

$L(s) = \vec{r}(t) + s(\vec{r}'(t))$

$\boxed{L(s) = \langle 0, 5, 0\rangle + s\langle -3, 0, -4\rangle}$

[redacted]

4. (10 points) Find the arc length, from $t = 0$ to $t = 1$ of the curve with parametrization $\mathbf{r}(t) = \langle t^2, t^3, 1 \rangle$.

$$S = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

$\|\vec{r}\,'(t)\|$

$$r(t) = \langle t^2, t^3, 1 \rangle$$

$$r'(t) = \langle 2t, 3t^2, 0 \rangle$$

$$\|r'(t)\| = \sqrt{(2t)^2 + (3t^2)^2 + 0^2}$$

$$\|r'(t)\| = \sqrt{4t^2 + 9t^4}$$

$$S = \int_0^1 \sqrt{4t^2 + 9t^4}\,dt$$

arc length $= \frac{1}{27}\left[(13)^{3/2} - 8\right]$

≈ 1.4397

$$S = \int_0^1 \sqrt{t^2(4 + 9t^2)}\,dt$$

$$S = \int_0^1 t\sqrt{4 + 9t^2}\,dt$$

$\sqrt{u} = u^{1/2}$

u-substitution

$u = 4 + 9t^2$

$du = 18t\,dt$

$\frac{1}{18}du = t\,dt$

$$S = \frac{1}{18}\int_0^1 \sqrt{u}\,du$$

$$S = \frac{1}{18}\left(\frac{2}{3}u^{3/2}\right)\Big]_0^1$$

$\sqrt[2]{4^3} = 8$

$$= \frac{1}{18}\left[\frac{2}{3}(4+9t^2)^{3/2}\right]_0^1 = \frac{1}{27}\left[(4+9)^{3/2} - (4+\right.$$

$$= \frac{1}{27}\left[(13)^{3/2} - (4)^{3/2}\right.$$

magnitude

5. (5 points) Show that if $\| \mathbf{r}(t) \| = 2$ then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. *(Hint: It might help you to recall that for any vector* $\mathbf{v}$ *we have that* $\| \mathbf{v} \|^2 = \mathbf{v} \cdot \mathbf{v}$*)*

Prove $r(t) \cdot r'(t) = 0$

$\|r(t)\|^2 = r(t) \cdot r(t) = 4$

$\|r(t)\| = 2 = \sqrt{4}$

$r(t) = 2$

$r'(t) = 0$

orthogonal vectors have a dot product of $\varnothing$

$2 \cdot 0 = 0$

$r(t) \cdot r'(t) = 0$

$r(t) \perp r'(t)$