Math 32A/1
Fall 2016
11/14/16
Time Limit: 50 Minutes

Name (Print): ______________________
SID Number: ______________________

Day \ T.A.	Alex	David	Kevin
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your answers. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) (a) Consider a particle moving in the plane:

$$\vec{r}(t) = \langle 3t - 4\sin(t), 1 - 4\cos(t)\rangle, \qquad 0 < t < 2\pi.$$

When does the particle attain its maximal speed?

(b) Find an arc length parametrization of $\vec{r}(t) = \langle -t^2, t^2, \frac{t^2}{2}\rangle$, $t \geq 0$. *Hint: Don't forget to specify the domain of your parametrization.*

Solution:

1. (4 points)

$$\vec{r}\,'(t) = \langle 3 - 4\cos(t), 4\sin(t)\rangle \qquad \text{(1 point)}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(3-4\cos(t))^2 + (4\sin(t))^2}$$

$$= \sqrt{9 - 24\cos(t) + 16\cos^2(t) + 16\sin^2(t)} = \sqrt{25 - 24\cos(t)} \qquad \text{(1 point)}$$

This attains its maximal value when $\cos(t) = -1$ (1 point), which occurs when $t = \pi$ (1 point).

Common mistakes:

- Basic arithmetic errors, such as people writing $\vec{r}\,'(t) = \langle 3 - 4\cos(t), 1 - 4\sin(t)\rangle$.
- Some people thought that $\sqrt{25 - 24\cos(t)}$ was minimized when $\cos(t) = 0$, but cos can take negative values.
- Finding the derivative of speed (which is the tangential component of acceleration) and solving for when that is zero is fine, but for future reference, you should also show that this point is a global maximum.
- Solving for when the acceleration vector is $\langle 0, 0\rangle$ (or equivalently, when its length is 0) is not correct, since the acceleration vector has a normal component which does not contribute to change in speed (also, the acceleration vector is never $\langle 0, 0\rangle$ in this problem).
- Solving for when the velocity vector is $\langle 0, 0\rangle$ (or equivalently, when the speed is 0) is also not correct, since if the speed was ever 0 (which it isn't), that would necessarily be the minimal speed, not maximal.
- The components of the velocity vector do not have to be maximized individually in order to maximize the speed, and in this problem, the components of the velocity vector cannot be maximized simultaneously anyway.

2. (6 points)

$$\vec{r}\,'(t) = \langle -2t, 2t, t\rangle \qquad \text{(1 point)}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-2t)^2 + (2t)^2 + t^2} = \sqrt{9t^2} = 3t \qquad \text{(1 point)}$$

$$s = \int_0^t \|\vec{r}\,'(x)\|\, dx = \int_0^t 3x dx = \frac{3}{2}t^2 \qquad \text{(1 point)}$$

$$t^2 = \frac{2}{3}s \qquad \text{(1 point)}$$

$$\vec{r_1}(s) = \langle -\frac{2}{3}s, \frac{2}{3}s, \frac{1}{3}s\rangle \qquad \text{(1 point)}$$

$\frac{2}{3}s = t^2$ for some $t \geq 0$ iff $s \geq 0$, so the domain is $s \geq 0$. (1 point)

Common mistakes:

- The most common mistakes were more arithmetic errors.
- A few people forgot to solve for t in $s = \frac{3}{2}t^2$, and made the substitution backwards, getting $\vec{r}_1(t) = \langle -(\frac{3}{2}t^2)^2, (\frac{3}{2}t^2)^2, \frac{(\frac{3}{2}t^2)^2}{2}\rangle$, or similar errors involving the substitution.
- The above mistakes could be checked by computing the speed of the parametrization, and checking to make sure that it is 1.
- Some people forgot to specify the domain, or gave an incorrect domain, such as $s \in \mathbb{R}$.

2. (10 points) (a) We saw in class that the curvature of a straight line is zero. Show the converse: if $\vec{r}(t) = \langle x(t), y(t) \rangle$ is a regular parametrization with zero curvature everywhere then the curve must be a straight line.
   (b) Consider the parametrization $\vec{r}(t) = \langle t, t^2 \rangle$. Write the acceleration vector $\vec{a}(\frac{1}{2})$ as the sum of vectors parallel and normal to the direction of motion.

Solution:

1. (4 points) There are many ways to solve this problem, for example as follows. $\|\vec{T}'(t)\| = \kappa(t) \cdot v(t) = 0$ for all t hence $\vec{T}(t)$ must be constant. This means that the tangent vector $\vec{v}(t) = v(t) \cdot \vec{T}(t)$ always points in the same direction. This can only happen if the curve is a line.

   Only few students gave a completely satisfactory answer here but many got partial credit. A typical attempt was to note that since
   $$0 = \kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3}$$
   (the cross product of vectors in the plane has to be interpreted corrrectly, by the way), $\vec{v} \times \vec{a}$ must be the zero vector. Some students seemed to believe erroneously that this implied both $\vec{v}$ and $\vec{a}$ had to be zero as well. However, what it *does* imply is that they are parallel, which means that the velocity vector doesn't change its direction and so we conclude again that the curve is a line.

2. (6 points) $\vec{v}(t) = \langle 1, 2t \rangle$ hence $\vec{v}(1/2) = \langle 1, 1 \rangle$. $\vec{a}(t) = \langle 0, 2 \rangle = \vec{a}(1/2)$. We want to write $\vec{a} = \vec{a}_{\|\vec{v}} + \vec{a}_{\perp\vec{v}}$ so we compute
   $$\vec{a}_{\|\vec{v}} = \frac{\vec{a} \cdot \vec{v}}{v^2}\vec{v} = \frac{\langle 0, 2 \rangle \cdot \langle 1, 1 \rangle}{1^2 + 1^2}\langle 1, 1 \rangle = \frac{2}{2}\langle 1, 1 \rangle = \langle 1, 1 \rangle.$$
   We must then have
   $$\vec{a}_{\perp\vec{v}} = \vec{a} - \vec{a}_{\|\vec{v}} = \langle 0, 2 \rangle - \langle 1, 1 \rangle = \langle -1, 1 \rangle,$$
   and we conclude that
   $$\vec{a}(1/2) = \langle 0, 2 \rangle = \langle 1, 1 \rangle + \langle -1, 1 \rangle,$$
   the first being the tangential component, the second the normal component.

   Many students had the correct idea but wasted a lot of time by computing the decomposition $\vec{a}(t) = \vec{a}_{\|\vec{v}}(t) + \vec{a}_{\perp\vec{v}}(t)$ for general t, substituting $t = 1/2$ only at the end of their calculations. This also led to many arithmetic mistakes being made.

3. (10 points) (a) Evaluate the limit or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} x^2 \sin(x/y^2).$$

(b) Show that the function

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}, \qquad (x,y,z) \neq (0,0,0),$$

satisfies the Laplace equation: $f_{xx} + f_{yy} + f_{zz} = 0$.

Solution:

(a) (5 points) Note that we have $0 \le |x^2 \sin(x/y^2)| \le x^2$ for all (x,y) with $y \neq 0$. As $\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} x^2 = 0$, we have that $\lim_{(x,y)\to(0,0)} |x^2 \sin(x/y^2)| = 0$ by the squeeze theorem, and it follows that $\lim_{(x,y)\to(0,0)} x^2 \sin(x/y^2) = 0$.

A number of students wrote that $0 \le |x^2 \sin(x/y^2)| \le x^2$ implies that

$$\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} |x^2 \sin(x/y^2)| \le \lim_{(x,y)\to(0,0)} x^2.$$

This isn't correct, as we don't a priori know that these limits exist. A number of students claimed that

$$\lim_{(x,y)\to(0,0)} x^2 \sin(x/y^2) = \left(\lim_{(x,y)\to(0,0)} x^2\right)\left(\lim_{(x,y)\to(0,0)} \sin(x/y^2)\right),$$

but this isn't true because the second limit in the product doesn't exist.

(b) (5 points) Let $g(x,y,z) = x^2+y^2+z^2$, so that $f = g^{-\frac{1}{2}}$. We then have that $f_x = -\frac{1}{2} g^{-\frac{3}{2}} g_x$, and

$$f_{xx} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) g^{-\frac{5}{2}}(g_x)^2 + (-\frac{1}{2}) g^{-\frac{3}{2}} g_{xx} = 3x^2 g^{-\frac{5}{2}} - g^{-\frac{3}{2}}.$$

Since f is symmetric in x, y, and z, we immediately see that

$$\begin{aligned} f_{yy} &= 3y^2 g^{-\frac{5}{2}} - g^{-\frac{3}{2}}, \text{ and} \\ f_{zz} &= 3z^2 g^{-\frac{5}{2}} - g^{-\frac{3}{2}}. \end{aligned}$$

Thus we have that

$$\begin{aligned} f_{xx} + f_{yy} + f_{zz} &= 3x^2 g^{-\frac{5}{2}} - g^{-\frac{3}{2}} + 3y^2 g^{-\frac{5}{2}} - g^{-\frac{3}{2}} + 3z^2 g^{-\frac{5}{2}} - g^{-\frac{3}{2}} \\ &= 3(x^2+y^2+z^2) g^{-\frac{5}{2}} - 3g^{-\frac{3}{2}} \\ &= (3g) g^{-\frac{5}{2}} - 3g^{-\frac{3}{2}} \\ &= 3g^{-\frac{3}{2}} - 3g^{-\frac{3}{2}} \\ &= 0 \end{aligned}$$

4. (10 points) (a) Consider the function $f(x,y) = \frac{1}{2x+y}$. Is the vector $\langle 1,-1,-1\rangle$ tangent to the graph $z = f(x,y)$ at the point $(\frac{1}{2}, 0, 1)$?

   (b) Use linear approximation to estimate $1.01^{3.99}$. *Hint:* $1.01^{3.99}$ *is defined as* $e^{3.99\ln(1.01)}$.

Solution:

1. (6 points) Yes. For this, we need to compute a normal vector to the graph of f at the given point, and show it's perpendicular to $\langle 1,-1,-1\rangle$. The normal vector can be given as $\langle -f_x(1/2,0), -f_y(1/2,0), 1\rangle$; this follows from the form of the tangent plane. $f_x = -2(2x+y)^{-2}$ and $f_y = -(2x+y)^{-2}$, so our normal vector becomes $\langle 2,1,1\rangle$. (A variation of the computation will give you $\langle -2,-1,-1\rangle$ instead, which is fine.) Indeed, $\langle 2,1,1\rangle \cdot \langle 1,-1,-1\rangle = 0$, so the vector is tangent, as we claimed.

   This is the most efficient solution, but it's also perfectly fine to write down the equation $z = 1 - 2(x-1/2) - y$ of the tangent plane to the graph of f at $(1/2,0)$ and see whether our vector lies in it. The trick is that you can *not* simply plug in $\langle 1,-1,-1\rangle$ to that equation! Instead you must use the vector equivalent to $\langle 1,-1,-1\rangle$ but based at $(1/2,0,1)$, namely, $\langle 3/2,-1,0\rangle$. Plugging the latter in to the equation of the tangent plane, we see the equation really does hold.

   A whole lot of people got 4 out of 6 points, here, most often for correctly writing down the tangent plane but failing to plug in the vector properly. There were a fair number of incorrect partial derivatives, as well, and a number of sign errors in writing down the normal vector.

   Exactly one person, I believe, bothered to check that f is actually differentiable, using the continuity of the partial derivatives, which was pretty awesome.

2. (4 points) The approximation is 1.04. We approximate using the function $f(x,y) = x^y$, which has $f_x = yx^{y-1}$ and $f_y = \ln(x)x^y$. In the natural base, we could also say $f(x,y) = e^{y\ln x}$, $f_x = \frac{y}{x}e^{y\ln x}$, $f_y = \ln(x)e^{y\ln x}$. It's easily checked that these are the same as the alternate forms given first. The approximation is

$$1.01^{3.99} = f(1.01, 3.99) \approx f(1,4) + .01 f_x(1,4) - .01 f_y(1,4) = 1 + .04 - 0 = 1.04,$$

   as claimed.

   There were a diverse array of errors here. First and foremost, a huge proportion of folks couldn't correctly write down the derivatives. It was extremely popular to write $f_x = \frac{1}{x}e^{y\ln x}$, for instance. Where'd the y go? Nobody knows! Plenty of people also missed the chain rule applications entirely and just wrote $f = f_x = f_y$. A word of advice: no one would ever put such a problem on a test. Anyway, this led to a wrong approximation, but you should have still gotten partial credit.

   Some people calculated Δf, instead of the approximation of the actual value of f. Some people got signs wrong, perhaps by using a formula instead of thinking things through.