Math 32A/1
Fall 2016
10/17/16
Time Limit: 50 Minutes

Name (Print): ______________________
SID Number: ______________________

Day \ T.A.	Alex	David	Kevin
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your answers. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Let $P = (4, 3)$, $Q = (2, -4)$. $O = (0, 0)$ denotes the origin.
   (a) Find the vector $\overrightarrow{PQ}$.
   (b) Find the unit vector in the direction of $\overrightarrow{OP}$. Simplify your answer.
   (c) Determine whether the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is obtuse ($> \pi/2$) or acute ($< \pi/2$).
   (d) Find the components of the vector
   $$4(2\hat{\imath} - 3\hat{\jmath}) - 2(3\hat{\imath} + \hat{\jmath}).$$
   (e) Determine whether the following three points are collinear, i.e. whether they lie on the same line:
   $$R = (-1, 2, 5), \quad T = (2, 4, 3), \quad U = (7, 1, 4).$$

Solution:

(a) (1 point) Just subtract the end point from the starting point:

$$\langle 2 - 4, -4 - 3\rangle = \langle -2, -7\rangle$$

(b) (2 points) Divide by the length. $||\overrightarrow{OP}|| = \sqrt{4^2 + 3^2} = 5$, so the appropriate unit vector is $\langle 4/5, 3/5\rangle$.

The only mistakes here came in incorrectly expressing the required vector or in dividing by the squared length instead of the length.

(c) (2 points) The dot product is $4 \cdot 2 + 3 \cdot (-4) = -4$. Thus the angle has negative cosine, which implies it's obtuse.

Various issues here, and some solutions that correctly used geometry instead of vector algebra. Not everyone knew how to identify obtuseness via the cosine, not everybody took the dot product of the right two vectors, many people calculated more than they needed to, and some of the latter calculations went wrong.

(d) (1 point) They're 2 and -14, respectively.

About the only issue here was sign errors.

(e) (4 points) They're not collinear, because the vectors $\overrightarrow{RT} = \langle 3, 2, -2\rangle$ and $\overrightarrow{RU} = \langle 8, -1, -1\rangle$ are not scalar multiples of each other. Indeed, a scalar multiplying with the former to give the latter would have to be both $8/3$ and $-1/2$!

Many people wasted a lot of time by calculating a cross product. This shows, to be fair, that you did the practice midterm, and works fine; but it's overkill, regardless. A majority of those who used the cross product technique calculated the cross product incorrectly, and a majority of *those* errors were sign errors. But virtually everybody at least correctly concluded that the cross product was nonzero.

2. (10 points) (a) Let $P = (1, 2, 3)$, $Q = (6, 5, 4)$, $R = (2, 3, 4)$. Compute the area of the triangle with vertices P, Q, R. Simplify your answer.

   (b) Calculate the components of
$$(2\hat{\imath} - 4\hat{\jmath}) \times (3\hat{k} + \hat{\jmath}).$$

Solution:

1. (7 points) We learned in class how to do this. Compute the vectors
$$\overrightarrow{PQ} = \langle 6-1, 5-2, 4-1\rangle = \langle 5, 3, 1\rangle \text{ (1 point)}$$
$$\overrightarrow{PR} = \langle 2-1, 3-2, 4-3\rangle = \langle 1, 1, 1\rangle \text{ (1 point)}$$

   and take the cross product of these:
$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \langle 5, 3, 1\rangle \times \langle 1, 1, 1\rangle \\ &= \langle 3\cdot 1 - 1\cdot 1, -5\cdot 1 + 1\cdot 1, 5\cdot 1 - 3\cdot 1\rangle \\ &= \langle 2, -4, 2\rangle\end{aligned}$$

   (2 points: 1 for knowing to compute the cross product, 1 for all the computation being correct).

   Finally the area is given by half the magnitude:
$$\begin{aligned}\frac{\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|}{2} &= \frac{\sqrt{2^2 + (-4)^2 + 2^2}}{2} \\ &= \frac{\sqrt{24}}{2} \\ &= \sqrt{6}.\end{aligned}$$

   (3 points: 1 for correct computation, 2 for correct formula. 1 point is given for formula that is similar to a correct formula).

   Common errors:

   - Using the dot product instead of cross product. The dot product of two vectors does not represent the area of a shape.
   - Some people used the dot product to get the cosine of the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and then tried to use that to get the sine of the angle, and then use $\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \left\|\overrightarrow{PQ}\right\| \left\|\overrightarrow{PR}\right\| \sin\theta$. This can be done correctly, and some people did do it correctly, but it is more difficult, and most people who tried it either made a mistake, or left their answer in an unsimplified form like $\frac{1}{2}\sqrt{105}\sin\left(\cos^{-1}\left(\frac{9}{\sqrt{105}}\right)\right)$, which is not an acceptable answer unless you actually evaluate that expression.
   - Adding together the coordinates of $\overrightarrow{PQ} \times \overrightarrow{PR}$ to get $2 - 4 + 2 = 0$. The cross product of two vectors is a vector, not a scalar.

- Forgetting to divide by 2. $\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|$ is the area of a parallelogram whose vertices include P, Q, and R, but you're looking for the area of a triangle.

2. (3 points)

$$\begin{aligned}\left(2\hat{i} - 4\hat{j}\right) \times \left(3\hat{k} + \hat{j}\right) &= 6\hat{i} \times \hat{k} - 12\hat{j} \times \hat{k} + 2\hat{i} \times \hat{j} - 4\hat{j} \times \hat{j} \\ &= -6\hat{j} - 12\hat{i} + 2\hat{k} + \overrightarrow{0} \\ &= \langle -12, -6, 2 \rangle .\end{aligned}$$

(1 point taken off per minor error.)

Common errors:

- Watch out for sign errors.
- Don't divide by a scalar. Some people attempted to simplify the answer to $\langle -6, -3, 1 \rangle$, which is a different vector.
- $\langle -12, -6, 2 \rangle$ and $-12\hat{i} - 6\hat{j} + 2\hat{k}$ are both fine, but $\left\langle -12\hat{i}, -6\hat{j}, 2\hat{k} \right\rangle$ is not a vector, because the components of a vector are scalars, and $\hat{i}$, $\hat{j}$, and $\hat{k}$ are vectors, not scalars.

3. (10 points) (a) Find the equation of the plane which contains the two lines parametrized by

$$\vec{r_1}(t) = \langle t, 1, 2t \rangle, \qquad \vec{r_2}(t) = \langle -1, -t, -2 \rangle.$$

(b) Find the equation of the plane through $P = (3, 4, 5)$ parallel to $x + 3y - z = 0$.

Solution:

1. (7 points) The lines have direction vectors $\vec{v_1} = \langle 1, 0, 2 \rangle$ and $\vec{v_2} = \langle 0, -1, 0 \rangle$. Since the plane containing these lines must contain these vectors,

$$\begin{aligned} \vec{v_1} \times \vec{v_2} &= \langle 0 \cdot 0 - 2 \cdot (-1), 1 \cdot 0 - 2 \cdot 0, 1 \cdot (-1) - 0 \cdot 0 \rangle \\ &= \langle 2, 0, -1 \rangle \end{aligned}$$

   is a normal vector to the plane. One point lying in the plane is $\vec{r_1}(0) = (0, 1, 0)$. Thus an equation for the plane is $2(x - 0) + 0(y - 1) + (-1)(z - 0) = 0$, which reduces to $2x - z = 0$.

   A number of students computed the cross products of the functions $\vec{r_1}(t)$ and $\vec{r_2}(t)$ and continued to use the resulting function throughout their solution. This caused the resulting equation for the plane to depend on t, which is incorrect.

2. (3 points) The normal vector to the plane $x + 3y - z = 0$ is $\langle 1, 3, -1 \rangle$, so an equation for such a plane is given by

$$1 \cdot (x - 3) + 3 \cdot (y - 4) + (-1) \cdot (z - 5) = 0$$

   which simplifies to

$$x + 3y - z = 10.$$

4. (10 points) (a) Parametrize the intersection of the surfaces $x^2 - z^2 = 2$ and $xy = 3$ using $x = t$ as parameter. Make sure to specify the domain of your parametrization.

(b) Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be non-zero vectors, and let λ be a scalar. Show that

$$\lambda(\overrightarrow{v}_{\parallel \overrightarrow{u}}) = (\lambda \overrightarrow{v})_{\parallel \overrightarrow{u}}.$$

Solution:

1. (7 points) Plugging in $x = t$ in the second equation we obtain

$$y = \frac{3}{t} \text{ if } t \neq 0. \tag{1}$$

Plugging in $x = t$ in the first equation we have $z^2 = t^2 - 2$ or

$$z = \pm\sqrt{t^2 - 2} \text{ if } |t| \geq \sqrt{2}. \tag{2}$$

Putting (1) and (2) together we obtain a parametrization of the intersection with two segments:

$$\overrightarrow{r_1}(t) = \langle t, \frac{3}{t}, \sqrt{t^2 - 2}\rangle$$
$$\overrightarrow{r_2}(t) = \langle t, \frac{3}{t}, -\sqrt{t^2 - 2}\rangle$$

with common domain $|t| \geq \sqrt{2}$. (Remark that $|t| \geq \sqrt{2}$ implies $t \neq 0$ so that $\frac{3}{t}$ is defined.)

A number of students solved correctly for y and z but gave a parametrization of the form $\overrightarrow{r}(t) = \langle t, \frac{3}{t}, \pm\sqrt{t^2 - 2}\rangle$ with $|t| \geq \sqrt{2}$. Note that this doesn't work since $\pm\sqrt{t^2 - 2}$ is not a function in t. Other students specified an incorrect domain.

2. (3 points) We start with the right-hand side of the equality

$$\begin{aligned}
(\lambda \overrightarrow{v})_{\parallel \overrightarrow{u}} &= \frac{(\lambda \overrightarrow{v}) \cdot \overrightarrow{u}}{\overrightarrow{u} \cdot \overrightarrow{u}} \overrightarrow{u} && \text{definition of the projection} \\
&= \frac{\lambda(\overrightarrow{v} \cdot \overrightarrow{u})}{\overrightarrow{u} \cdot \overrightarrow{u}} \overrightarrow{u} && \text{rule for the dot product} \\
&= \lambda \frac{\overrightarrow{v} \cdot \overrightarrow{u}}{\overrightarrow{u} \cdot \overrightarrow{u}} \overrightarrow{u} \\
&= \lambda(\overrightarrow{v}_{\parallel \overrightarrow{u}}) && \text{definition of the projection}
\end{aligned}$$

and end up with the left-hand side of the equality.

Some students expressed $\overrightarrow{u}$ and $\overrightarrow{v}$ in components and simply computed both sides of the equality. This is perfectly fine but takes longer.