Math 32A/1
Fall 2016
12/8/16
Time Limit: 180 Minutes

Name (Print): ____________________
SID Number: ____________________

Day \ T.A.	Alex	David	Kevin
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

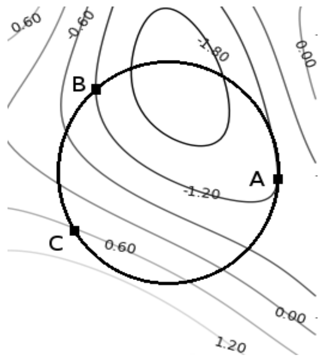
1. (10 points) (a) Estimate the magnitude of the vector $\langle 1.03, 1.99, 2.01\rangle$ using linear approximation.

   (b) Evaluate the limit or show that it does not exist: $\lim_{(x,y)\to(0,0)} \frac{x+y}{\sqrt{x^2+y^2}}$.

Answer key:

1. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $f_x = x/f$, $f_y = y/f$, $f_z = z/f$. $f(1.03, 1.99, 2.01) \approx f(1,2,2) + 0.03 f_x - 0.01 f_y + 0.01 f_z = 3 + 0.03 \cdot 1/3 - 0.01 \cdot 2/3 + 0.01 \cdot 2/3 = 3.01$.
2. $y = mx$, $\frac{|x+y|}{\sqrt{x^2+y^2}} = \frac{|1+m|}{\sqrt{1+m^2}}$ depends on m hence the limit does not exist.

2. (10 points) (a) The figure below shows the level curves of a differentiable function $f(x, y)$ and a circular path $\overrightarrow{r}(t)$, traversed in counterclockwise direction. State whether the derivative $\frac{d}{dt} f(\overrightarrow{r}(t))$ is positive, negative, or zero at each of the points A, B, C.



(b) Find the points P on the surface $z^2 - z = 2x^2 + 3y^2$ such that the tangent plane at P contains the origin.

Answer key:

1. $A := 0$, $B :> 0$, $C :> 0$.
2. equation for tangent plane at (a, b, c): $-4a(x-a) - 6b(y-b) + (2z-1)(z-c) = 0$. Origin lies on the plane if and only if $4a^2 + 6b^2 = 2c^2 - c$. But also $2a^2 + 3b^2 = c^2 - c$ so $c = 0$ hence $a = b = c = 0$.

3. (10 points) (a) Let $g(x,y) = (x-y)^2 + \frac{x}{y}$, and let (r,θ) be polar coordinates. Evaluate $\frac{\partial g}{\partial \theta}$ and $\frac{\partial g}{\partial r}$ at $(x,y) = (0,1)$.

   (b) Find all local minima of $f(x,y) = \frac{x^3}{3} - yx + y^2$.

Answer key:

1. $g_x = 2(x-y) + 1/y$, $g_x(0,1) = -1$; $g_y = 2(y-x) - x/y^2$, $g_y(0,1) = 2$. $g_\theta = g_x x_\theta + g_y y_\theta = 1$. $g_r = g_x x_r + g_y y_r = 2$.

2. $f_x = x^2 - y$, $f_y = -x + 2y$. Critical points $y = x^2$, $x = 2y$ so $(0,0)$ and $(1/2, 1/4)$. $f_{xx} = 2x$, $f_{yy} = 2$, $f_{xy} = -1$ hence $D(0,0) = -1$, $D(1/2,1/4) = 1$, $f_{xx}(1/2,1/4) = 1 > 0$. Local minimum at $(1/2,1/4)$.

4. (10 points) The temperature at time t at the point (x, y) is given by $T(x, y, t) = y(x - 1 + t)$.
   (a) What is the maximum temperature occuring on the square $0 \le x, y \le 1$ at time $t = 0$?
   (b) What is the minimum temperature subject to the constraints $x^2 + y^2 = 1$ and $t = 1$?

Answer key:

1. $f(x, y) = T(x, y, 0) = (x - 1)y \le 0$ for all $0 \le x, y \le 1$ and $f(1, 1) = 0$ so maximum is 0.
2. $f(x, y) = T(x, y, 1) = xy$. $g(x, y) = x^2 + y^2 - 1$. $\nabla f = \langle y, x\rangle$, $\nabla g = 2\langle x, y\rangle$. Lagrange condition $y = \lambda x$, $x = \lambda y$. 4 solutions satisfying $g(x, y) = 0$: $1/\sqrt{2}(\pm 1, \pm 1)$. Minimum $-1/2$ since constraint is closed and bounded.

5. (10 points) (a) A bird flying east at $\sqrt{2} \cdot 5$ m/h encounters a 5-m/h wind blowing in the north-west direction. What is the resultant speed of the bird? *Hint: Your answer should be an integer in m/h.*

   (b) Determine whether the points $P = (0, 1, 2)$, $Q = (1, 1, 1)$, $R = (-1, -1, -1)$, $S = (1, -2, 3)$ all lie on a common plane.

Answer key:

1. $\overrightarrow{v_{\text{bird}}} = 5\sqrt{2}\langle 1, 0\rangle$, $\overrightarrow{v_{\text{wind}}} = 5/\sqrt{2}\langle -1, 1\rangle$. $\overrightarrow{v} = \overrightarrow{v_{\text{bird}}} + \overrightarrow{v_{\text{wind}}} = 5/\sqrt{2}\langle 1, 1\rangle$, $v = 5$
2. $\overrightarrow{PQ} = \langle 1, 0, -1\rangle$, $\overrightarrow{PR} = \langle -1, -2, -3\rangle$, $\overrightarrow{n} := \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -2, 4, -2\rangle$, $\overrightarrow{n} \cdot \overrightarrow{OP} = 0$; equation of plane containing P, Q, R: $\overrightarrow{n} \cdot \langle x, y, z\rangle = 0$. $\overrightarrow{n} \cdot \overrightarrow{OS} = -16 \neq 0$ hence S doesn't lie on the plane. The answer is no.

6. (10 points) (a) Suppose $\overrightarrow{r}(t)$ and $\overrightarrow{s}(t)$ are two parametrizations such that for all t, $\overrightarrow{r}(t)$ is orthogonal to $\overrightarrow{s}(t)$. Show that $\overrightarrow{r}'(t) \cdot \overrightarrow{s}(t) = -\overrightarrow{r}(t) \cdot \overrightarrow{s}'(t)$.

   (b) Let $\mathcal{C}$ be the curve defined as the intersection of $x = 2e^y$ and $x^2 = 4z$. Compute the length of the segment of $\mathcal{C}$ defined by $0 \leq y \leq 1/2$.

Answer key:

1. $0 = \overrightarrow{r} \cdot \overrightarrow{s}$ hence $0 = (\overrightarrow{r} \cdot \overrightarrow{s})' = \overrightarrow{r}' \cdot \overrightarrow{s} + \overrightarrow{r} \cdot \overrightarrow{s}'$ by the dot product rule. The claim follows.
2. $y = t$, $x = 2e^t$, $z = e^{2t}$. $\overrightarrow{v} = \langle 2e^t, 1, 2e^{2t} \rangle$, $v = 1 + 2e^{2t}$. $\int_0^{1/2} v \cdot dt = e - 1/2$.

7. (10 points) A particle is tracing the curve $y = x^2$ from left to right at constant speed 3.
   (a) Find the curvature κ at the point $(-1, 1)$.
   (b) Find the velocity vector $\overrightarrow{v}$ of the particle at the point $(-1, 1)$.
   (c) Write the acceleration vector $\overrightarrow{a}$ at the point $(-1, 1)$ as a sum of vectors parallel and normal to the direction of motion.

Answer key:

1. $\overrightarrow{r_1}(t) = \langle t, t^2 \rangle$, $\overrightarrow{v_1}(t) = \langle 1, 2t \rangle$, $\overrightarrow{a_1}(t) = \langle 0, 2 \rangle$. $\kappa(-1) = \frac{\|\overrightarrow{v_1}(-1) \times \overrightarrow{a_1}(-1)\|}{v_1(-1)^3} = \frac{2}{5^{3/2}}$.
2. $\overrightarrow{v} = \frac{3}{v_1(-1)} \overrightarrow{v_1}(-1) = \frac{3}{\sqrt{5}} \langle 1, -2 \rangle$.
3. $\overrightarrow{N} \perp \overrightarrow{v_1}$ pointing inwards so $\overrightarrow{N} = \frac{\langle 2,1 \rangle}{\|\langle 2,1 \rangle\|} = \frac{\langle 2,1 \rangle}{\sqrt{5}}$. $a_N = \kappa \cdot v^2 = \frac{2 \cdot 3^2}{5^{3/2}}$, $a_T = 0$. $\overrightarrow{a} = \frac{2 \cdot 3^2}{5^2} \langle 2, 1 \rangle + \overrightarrow{0}$.

8. (10 points) (a) Let $g(x, y, z) = 12e^{x^2(y-1)} + \sin(z - x)y$. Compute g_{xxzy}.

   (b) Suppose $f(x, y)$ is differentiable and let $\overrightarrow{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$ and $\overrightarrow{v} = \frac{1}{\sqrt{2}}\langle 1, -1\rangle$. Show that if $D_{\overrightarrow{u}} f(0, 0) = D_{\overrightarrow{v}} f(0, 0) = 0$, then $f_x(0, 0) = 0$.

Answer key:

1. $g_{xxzy} = g_{zyxx} = -\cos(z - x)$.
2. $f_x(0, 0) = D_{\langle 1,0\rangle} f(0, 0) = \nabla f(0, 0) \cdot \langle 1, 0\rangle = \nabla f(0, 0) \cdot \frac{1}{\sqrt{2}}(\overrightarrow{u} + \overrightarrow{v}) = \frac{1}{\sqrt{2}}\nabla f(0, 0) \cdot \overrightarrow{u} + \frac{1}{\sqrt{2}}\nabla f(0, 0) \cdot \overrightarrow{v} = \frac{1}{\sqrt{2}}D_{\overrightarrow{u}} f(0, 0) + \frac{1}{\sqrt{2}}D_{\overrightarrow{v}} f(0, 0) = 0$.