UCLA

Department of Mathematics

Math 32A

Sections 2&4 Fall 2016

Midterm #1

Before the exam is opened, fill out the bottom portion of the front. A few items of interest: (1) It will be seen that the point–value of the problems does not necessarily reflect their difficulty. Thus make sure that you actually have correct answers written down for the problems you know how to do. (2) The course, so far, and this exam has been vector based. For these problems, always try a vector based derivation; only resort to old–fashioned methods if desperate. (3) You will notice that the point values of the problems total up to something more than 100. Nevertheless, it is not possible to do better than 100% on this exam. Here's the deal: Do all the problems, you will get to keep all the points that you earn unless/until your total exceeds 100. Then you simply (max out and) get 100%. Good luck, really do take your time with the problems and check over your answers.

The following formula is famous and probably correct:
$\vec{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\vec{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Then

$$\vec{A} \times \vec{B} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Question #1	20 / 20
Question #2	20 / 20
Question #3	15 / 15

Question #4	15 / 15
Question #5	15 / 15
Question #6	14 / 15
Question #7	7 / 15

Total	100 / 100

Name: Aggarwal Vaibhav ____
(Please Print) – Last – – First – M.I.

Section: 2, B

Student ID Number: 1 0 4 – 7 6 8 – 1 2 7

hidden

Question (1) (20 Points). Let $\vec{q} = \langle a, b\rangle$ denote a (non–zero) two–dimensional vector

Part A (10 points). Find the vector – which is unique up to a scalar factor of ± 1 – that is orthogonal to $\vec{s}$ and has the same length as $\vec{s}$. You must justify your *answer* (but you need not justify your derivation).

orthogonal if $\vec{q} \cdot \vec{v} = 0$

$\boxed{\vec{v} = \langle -b, a \rangle}$ $\langle a, b\rangle \cdot \langle -b, a\rangle = -ab + ab = 0$ ✓ orthogonal

$\|\vec{q}\| = \sqrt{a^2 + b^2}$

$\|\vec{v}\| = \sqrt{b^2 + a^2}$ ✓ same length

10

Part B (10 points). Let $\vec{p} = \langle \alpha, \beta, \gamma\rangle$ denote a (non–zero) 3D vector and $\lambda > 0$ a positive number. Find the vector which points in the same direction as $\vec{p}$ but has length λ.

$$\hat{p} = \frac{\langle \alpha, \beta, \gamma \rangle}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

10

$$\boxed{\vec{v} = \lambda \left\langle \frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}, \frac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}, \frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \right\rangle} \boxed{= \lambda \hat{p}}$$

20

Question (2) (20 Points) Let $\mathbf{I}$ and $\mathbf{J}$ denote the *un*standard unit vectors

$$\mathbf{I} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta, \quad \mathbf{J} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

5

Part A (5 points). Compute $||\mathbf{I}||$, $||\mathbf{J}||$ and $\mathbf{I}\cdot\mathbf{J}$.

$$||I|| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$||J|| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$I \cdot J = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

10

Part B (10 points). Let $\langle a, b\rangle$ denote an arbitrary 2D vector. We wish to write $\langle a, b\rangle = \alpha\mathbf{I} + \beta\mathbf{J}$. Find expressions for α and β in terms of a, b and θ.

$$\vec{V} = \langle a, b\rangle$$

$$\text{Proj}_I \vec{V} + \text{proj}_J \vec{V} = (\vec{V}\cdot I)I + (\vec{V}\cdot J)J$$

$$\alpha = \vec{V}\cdot I = a\cos\theta + b\sin\theta$$

$$\beta = \vec{V}\cdot J = -a\sin\theta + b\cos\theta$$

5

Part C (5 points). Now let us regard $\mathbf{I}$ and $\mathbf{J}$ as three component vectors whose third component happens to be zero. E.g., now, $\mathbf{I} = \langle\cos\theta, \sin\theta, 0\rangle$. Compute $\mathbf{I}\times\mathbf{J}$.

$-\sin\theta \quad \cos\theta \quad 0$

$$I \times J = \langle 0, 0, \cos^2\theta + \sin^2\theta\rangle = \boxed{\langle 0, 0, 1\rangle}$$

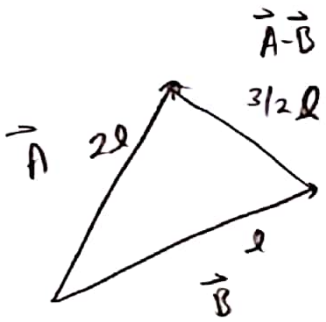
Question (3) (15 Points). Let $\vec{A}$ be a vector in $\mathbb{R}^2$ which has length 2ℓ. Let $\vec{B}$, also a vector in $\mathbb{R}^2$, have a length of ℓ. It is known that the distance between (the tips of) $\vec{A}$ and $\vec{B}$ is $\frac{3}{2}\ell$. On this basis, compute the dot product $\vec{A} \cdot \vec{B}$. Hint: Write an expression for (the square of) the distance between the tips.

$\vec{A} = \langle a_1, a_2 \rangle$

$\vec{B} = \langle b_1, b_2 \rangle$

$\|\vec{A}\| = \sqrt{a_1^2 + a_2^2} = 2\ell$

$\|\vec{B}\| = \sqrt{b_1^2 + b_2^2} = \ell$



$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2$

$$\|\vec{A}-\vec{B}\|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$\frac{9}{4}\ell^2 = a_1^2 + b_1^2 - 2a_1 b_1 + a_2^2 + b_2^2 - 2a_2 b_2$$

$$\frac{9}{4}\ell^2 = (2\ell)^2 + \ell^2 - 2a_1 b_1 - 2a_2 b_2$$

$$\frac{9}{4}\ell^2 - \frac{20}{4}\ell^2 = -2(a_1 b_1 + a_2 b_2)$$

$$\boxed{\frac{11}{8}\ell^2 =} \; a_1 b_1 + a_2 b_2 = \vec{A} \cdot \vec{B}$$

Question (4) (15 Points). Consider the plane in $\mathbb{R}^3$ which is given by

$$z = \alpha x + \beta y$$

with both α and β non–zero. This plane intersects the xz–plane forming some line in $\mathbb{R}^3$. Find, in vector form, the equation for this line of intersection.

$P_1 \rightarrow \alpha x + \beta y - z = 0$

$P_2 \rightarrow y = 0$

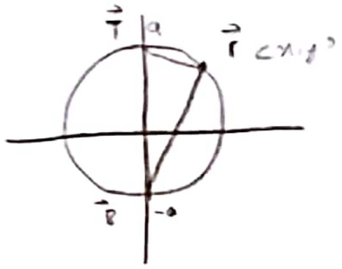
$\vec{r}_0$ = point in both planes $= \langle 1, 0, \alpha \rangle$

$\vec{V} = \vec{n}_1 \times \vec{n}_2 = \langle \alpha, \beta, -1 \rangle \times \langle 0, 1, 0 \rangle = \langle 1, 0, \alpha \rangle$

$$\boxed{\vec{r} = \langle 1, 0, \alpha \rangle + \langle 1, 0, \alpha \rangle t, \ -\infty < t < \infty}$$

Question (5) (15 Points). Let C_a denote the circle of radius a centered at the origin and consider the "top" and "bottom" points of this circle i.e., $\langle 0, \pm a \rangle$. Show that these two points on C_a along with any other (arbitrary) point on C_a always form a right triangle.

$\vec{T}$ = top point = $\langle 0, a \rangle$
$\vec{B}$ = bottom point = $\langle 0, -a \rangle$
$\vec{r}$ = point on triangle = $\langle x, y \rangle$



$\|\vec{r}\| = a = \sqrt{x^2+y^2}$

$(\vec{r}-\vec{T})\cdot(\vec{r}-\vec{B})$

$= \vec{r}\cdot\vec{r} - \vec{r}\cdot\vec{T} - \vec{r}\cdot\vec{B} + \vec{T}\cdot\vec{B}$

$= x^2+y^2 - ay + ay - a^2 = x^2+y^2 - \sqrt{x^2+y^2}^{\,2} = 0$ ✓

orthogonal
∴ right triangle

Question (6) (15 Points). Consider the vector $\vec{\beta} = \frac{1}{\sqrt{2}}\langle b, b\rangle$ with $b > 0$ and, for $a < b$ consider the circle of radius a centered at the point $\vec{\beta}$:

$$C = \{\vec{r} \in \mathbb{R}^2 \mid ||\vec{\beta} - \vec{r}|| = a\}.$$

Part A (5 points). Write a parametric description, e.g., $\langle x(t), y(t)\rangle$, for the curve C.

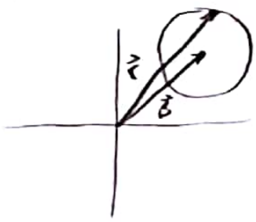
$$\boxed{\begin{aligned} x(t) &= \tfrac{1}{\sqrt{2}}b + a\cos t \\ y(t) &= \tfrac{1}{\sqrt{2}}b + a\sin t \end{aligned}}$$

$$\boxed{< \tfrac{1}{\sqrt{2}}b + a\cos t,\ \tfrac{1}{\sqrt{2}}b + a\sin t >,\quad 0 \le t \le 2\pi}$$

Part B (10 points). Using the above, find out which point on C is nearest and which point on C is farthest from the origin and what are these distances. Full derivation is required for full credit.

Hint: As always, remember it is usually best to work with the squares of distances.

$C = \{\vec{r} \in \mathbb{R}^2 \mid ||\vec{\beta} - \vec{r}|| = a\}$



$\vec{r} = < \frac{1}{\sqrt{2}}b + a\cos t,\ \frac{1}{\sqrt{2}}b + a\sin t >,\ 0 \le t \le 2\pi$

$||\vec{r}|| = \sqrt{\frac{1}{2}b^2 + a^2\cos^2 t + \sqrt{2}\,ba\cos t) + \frac{1}{2}b^2 + a^2\sin^2 t + \sqrt{2}\,ab\sin t}$

$= \sqrt{b^2 + a^2 + ab\sqrt{2}\cos t + ab\sqrt{2}\sin t}$

$\dfrac{d||\vec{r}||}{dt} = \dfrac{-ab\sqrt{2}\sin t + ab\sqrt{2}\cos t}{2\sqrt{b^2 + a^2 + ab\sqrt{2}\cos t + ab\sqrt{2}\sin t}} = 0$

$\cos t - \sin t = 0$

$\cos t = \sin t$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \cancel{\frac{7\pi}{4}}$

$||\vec{r}(\frac{\pi}{4})|| = \sqrt{b^2 + a^2 + 2ab} = a + b$ $\quad ||\vec{r}(\frac{5\pi}{4})|| = \sqrt{b^2 + a^2 - 2ab} = b - a$

$||\vec{r}(\frac{3\pi}{4})|| = \sqrt{b^2 + a^2}$ $\quad ||\vec{r}(\frac{7\pi}{4})|| = \sqrt{b^2 + a^2}$

no absolute value needed since $a < b$

$\boxed{\begin{array}{l} \text{farthest } @\ t = \frac{\pi}{4} \\ \text{to origin} \\ \text{closest } @\ t = \frac{5\pi}{4} \\ \text{to origin} \end{array}}$

$\boxed{\begin{array}{l} \text{farthest} \to < \frac{1}{\sqrt{2}}b + \frac{a\sqrt{2}}{2}, \frac{1}{\sqrt{2}}b + \frac{a\sqrt{2}}{2} > \\ \text{closest} \to < \frac{1}{\sqrt{2}}b - \frac{a\sqrt{2}}{2}, \frac{1}{\sqrt{2}}b - \frac{a\sqrt{2}}{2} > \end{array}}$

$\boxed{\begin{array}{l} \text{farthest is } a+b \text{ units away from origin} \\ \text{closest is } b-a \text{ units away from origin} \end{array}}$

Question (7) (15 Points). Consider the plane $\mathbb{P}$ defined by $x + y + z = 0$.

Part A (5 points). What is the normal to this plane?

$\vec{n} = \langle 1, 1, 1 \rangle$

5

Part B (5 points). Find two orthonormal vectors which lie in this plane.

$\vec{r}_1 = \langle 1, -2, 1 \rangle \rightarrow \vec{r}_1 \cdot \vec{n} = 1-2+1 = 0$ ✓

$\vec{r}_2 = \langle 2, -1, -1 \rangle \rightarrow \vec{r}_2 \cdot \vec{n} = 2-1-1 = 0$ ✓

both vectors orthogonal to normal & lie in plane $x+y+z=0$

2

But $\vec{r}_1 \cdot \vec{r}_2 \neq 0$

Part C (5 points). Using your result from Part B find a parametric representation for a circle of radius a which lies in the plane $\mathbb{P}$.

$\cancel{\sqrt{(x-1)^2 + (y+2)^2 + (z+1)^2} = a}$

$$\boxed{\begin{aligned} x(t) &= 1 + a\cos t \\ y(t) &= -2 + a\sin t \\ z(t) &= 1 - a\cos t - a\sin t \end{aligned}}$$