1. (a) (10 points) Consider $\mathbf{r}(t) = \langle 2t-2, t^3+1\rangle$. Find the parametric equation of a tangent line at $t=1$.

$r'(t) = \langle x'(t), y'(t)\rangle$

$r(1) = \langle 2(1)-2, (1)^3+1\rangle$
$= \langle 2-2, 1+1\rangle$
$= \langle 0, 2\rangle$

$r'(t) = \langle 2, 3t^2\rangle$

$r'(1) = \langle 2, 3(1)^2\rangle$
$= \langle 2, 3\rangle$

$\boxed{L(t) = \langle 0, 2\rangle + t\langle 2, 3\rangle}$

(b) (10 points) Find the area under the curve $c(t) = \langle 2t-2, t^3+1\rangle$ for $2 \le t \le 3$.

$\text{Area} = \int_{t_1}^{t_2} y(t)\,x'(t)\,dt$

$\text{Area} = \int_2^3 (t^3+1)(2)\,dt$
$= \int_2^3 (2t^3+2)\,dt$
$= (\frac{1}{2}t^4 + 2t)\Big|_2^3$
$= (\frac{1}{2}(3)^4+2(3)) - (\frac{1}{2}(2)^4+2(2))$
$= (\frac{1}{2}(81)+6) - (\frac{1}{2}(16)+4)$
$= (40.5+6) - (8+4)$
$= 46.5 - 12$
$= \boxed{34.5}$

1.1 (a) **10 / 10**

✓ **+ 10 pts** **Full credit; using the formula $$$L(s) = \vec{r}(t) + s\vec{r}'(t)$$$ we get that the tangent line is $$$L(s) = \langle 0, 2 \rangle + s \langle 2, 3 \rangle = \langle 2s, 2 + 3s \rangle$$$.**

(One can of course use any variable name in the tangent line, not just $$s$$, or give any equivalent parametric form of the line.).

- **+ 9 pts** Correct up to purely computational error.
- **+ 9 pts** The line is correct, but it is not in parametric form as requested by the problem.

(In class, did the point-slope form which would be $$y = (3/2)x + 2$$, but that isn't a parametric form; you could write something like $$x = t, y = (3/2)t + 2$$).

- **+ 8 pts** Same as the (9pt) rubric item above, but computational error.
- **+ 8 pts** Error that is slightly more than computational (e.g. forgets to plug in $$t = 1$$)
- **+ 5 pts** Finds $$\frac{dy}{dx}$$ or $$\vec{r}'(t)$$ correctly but doesn't give a correct tangent line

1. (a) (10 points) Consider $\mathbf{r}(t) = \langle 2t-2, t^3+1 \rangle$. Find the parametric equation of a tangent line at $t = 1$.

$r'(t) = \langle x'(t), y'(t) \rangle$

$$r(1) = \langle 2(1)-2, (1)^3+1 \rangle$$
$$= \langle 2-2, 1+1 \rangle$$
$$= \langle 0, 2 \rangle$$
$$r'(t) = \langle 2, 3t^2 \rangle$$
$$r'(1) = \langle 2, 3(1)^2 \rangle$$
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$$\boxed{L(t) = \langle 0, 2 \rangle + t \langle 2, 3 \rangle}$$

(b) (10 points) Find the area under the curve $c(t) = \langle 2t-2, t^3+1 \rangle$ for $2 \le t \le 3$.

$$\text{Area} = \int_{t_1}^{t_2} y(t)\, x'(t)\, dt$$

$$\text{Area} = \int_2^3 (t^3+1)(2)\, dt$$
$$= \int_2^3 (2t^3+2)\, dt$$
$$= \left(\tfrac{1}{2}t^4 + 2t\right)\Big|_2^3$$
$$= \left(\tfrac{1}{2}(3)^4 + 2(3)\right) - \left(\tfrac{1}{2}(2)^4 + 2(2)\right)$$
$$= \left(\tfrac{1}{2}(81)+6\right) - \left(\tfrac{1}{2}(16)+4\right)$$
$$= (40.5+6) - (8+4)$$
$$= 46.5 - 12$$
$$= \boxed{34.5}$$

1.2 (b) **10 / 10**

✓ **+ 10 pts Correct; the formula $$$A = \int_{a}^b y(t) x'(t) \ dt$$$ gives $$$A = \int_2^3 (t^3 + 1)(2) \ dt = \frac{69}{2}$$$**

+ 9 pts Correct up to computational error.

+ 8 pts Correct up to small mathematical error (i.e. more than computational) involving the bounds of integration

+ 2 pts Integrated both coordinates; this would give the absement (https://en.wikipedia.org/wiki/Absement) because we are integrating displacement. It is not relevant to the area under the curve.

Note that this gives a _vector_ $$\langle 3, 69/4\rangle$$ when we are looking for a _number_.

2. (20 points) Compute the (acute) angle of intersection of two planes $x + z = 2$ and $2x - y - z = 1$ (express the angle using the arccos function).

$n_1 = \langle 1, 0, 1 \rangle$

$n_2 = \langle 2, -1, -1 \rangle$

$$\cos\theta = \frac{n_1 \cdot n_2}{\|n_1\| \|n_2\|}$$

$\|n_1\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$\|n_2\| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{4+1+1} = \sqrt{6}$

$$\begin{aligned} n_1 \cdot n_2 &= \langle 1, 0, 1 \rangle \cdot \langle 2, -1, -1 \rangle \\ &= (1 \times 2) + (0 \times -1) + (1 \times -1) \\ &= 2 + 0 - 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{1}{\sqrt{2}\sqrt{6}} \\ &= \frac{1}{\sqrt{12}} \\ &= \frac{1}{2\sqrt{3}} \end{aligned}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{2\sqrt{3}}\right)}$$

2 Problem 2 **20 / 20**

- ✓ **+ 5 pts** Extracted normal vectors correctly
- ✓ **+ 5 pts** Correct lengths
- ✓ **+ 5 pts** Correct dot product
- ✓ **+ 5 pts** Correct formula for angle
- **- 2 pts** Minor error (with work shown)

3. (20 points) Consider two vectors $\mathbf{v}$ and $\mathbf{w}$ such that

$$\|\mathbf{v}\| = 2, \quad \|\mathbf{w}\| = 1, \quad \|\mathbf{v}+\mathbf{w}\| = \sqrt{3}.$$

Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

$\cos\theta = \frac{v \cdot w}{\|v\| \, \|w\|} = \frac{v \cdot w}{2}$ relation with length : $v \cdot v = \|v\|^2$

$$\|v+w\|^2 = (v+w)\cdot(v+w)$$
$$= v\cdot(v+w) + w\cdot(v+w)$$
$$= v\cdot v + v\cdot w + v\cdot w + w\cdot w$$
$$= \|v\|^2 + 2(v\cdot w) + \|w\|^2$$
$$= 2^2 + 2(v\cdot w) + 1^2$$
$$\sqrt{3} = 5 + 2(v\cdot w)$$
$$3-5 = 2(v\cdot w)$$
$$-2 = 2(v\cdot w)$$
$$(v\cdot w) = -1$$

$$\therefore \cos\theta = \frac{-1}{2}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-1}{2}\right)}$$

3 Problem 3 **20 / 20**

- ✓ **+ 5 pts** **Considered square magnitude of v+w (or had square roots on the other side)**
- ✓ **+ 5 pts** **Expanded dot product correctly**
- ✓ **+ 5 pts** **Solved for dot product of v,w or wrote in cos formula**
- ✓ **+ 5 pts** **Correct formula for angle**
- **+ 20 pts** did something else (e.g. law of cosines, 30/60/90 triangle) correctly
- **- 2 pts** Minor error (with work)
- **+ 15 pts** Tried something else (e.g. law of cosines, 30/60/90 triangle) but made a conceptual error, or did not correctly set something up (could get right answer by mistake this way, since law of cosines applied correctly gives you the wrong angle at first)
- **- 2 pts** Found wrong angle (e.g. angle between negative v and w), or gave more than one angle
- **+ 10 pts** Some work looks correct but hard to follow
- **+ 0 pts** No credit

4. Consider three points $P = (1, 0, -1)$, $Q = (0, -2, 1)$ and $R = (2, -1, 0)$.

(a) (10 points) Find the area of triangle formed by points P, Q, and R.

$$\text{area} = \frac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \|$$

$$\overrightarrow{PQ} = \langle 0-1, -2-0, 1-(-1) \rangle = \langle -1, -2, 2 \rangle$$

$$\overrightarrow{PR} = \langle 2-1, -1-0, 0-(-1) \rangle = \langle 1, -1, 1 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & -2 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ -1 & 1 \end{vmatrix} i - \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} k$$

$$= \langle -2-(-2), -(-1-2), 1-(-2) \rangle = \langle 0, 3, 3 \rangle$$

$$\| \overrightarrow{PQ} \times \overrightarrow{PR} \| = \sqrt{0^2 + 3^2 + 3^2} = \sqrt{18}$$

$$\text{area} = \frac{1}{2}(\sqrt{18}) = \sqrt{18/4} = \sqrt{9/2} = \boxed{3\sqrt{\tfrac{1}{2}} \text{ units}^2}$$

(b) (10 points) Find the equation of the plane containing points P, Q, and R.

$$n = \overrightarrow{PQ} \times \overrightarrow{PR}$$

$$= \langle 0, 3, 3 \rangle$$ (*as per above work)

* P will be reference point

$$0(x-1) + 3(y-0) + 3(z-(-1)) = 0$$

$$\boxed{3y + 3(z+1) = 0}$$

4.1 (a) **10 / 10**

- ✓ **- 0 pts** Correct
- **- 3 pts** wrong vector
- **- 1 pts** too complicated. write solution to the simplest form
- **- 2 pts** wrong cross product
- **- 2 pts** wrong area
- **- 1 pts** final answer wrong
- **- 3 pts** wrong way to calculate cross product
- **- 1 pts** cross product sign wrong

4. Consider three points $P = (1, 0, -1)$, $Q = (0, -2, 1)$ and $R = (2, -1, 0)$.

(a) (10 points) Find the area of triangle formed by points P, Q, and R.

$$\text{area} = \tfrac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \|$$

$$\overrightarrow{PQ} = \langle 0-1, -2-0, 1-(-1) \rangle = \langle -1, -2, 2 \rangle$$

$$\overrightarrow{PR} = \langle 2-1, -1-0, 0-(-1) \rangle = \langle 1, -1, 1 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & -2 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ -1 & 1 \end{vmatrix} i - \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} k$$

$$= \langle -2-(-2), -(-1-2), 1-(-2) \rangle = \langle 0, 3, 3 \rangle$$

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$$\text{area} = \tfrac{1}{2}(\sqrt{18}) = \sqrt{18/4} = \sqrt{9/2} = \boxed{3\sqrt{\tfrac{1}{2}} \text{ units}^2}$$

(b) (10 points) Find the equation of the plane containing points P, Q, and R.

$$n = \overrightarrow{PQ} \times \overrightarrow{PR}$$

$$= \langle 0, 3, 3 \rangle \quad (\text{*as per above work})$$

* P will be reference point

$$0(x-1) + 3(y-0) + 3(z-(-1)) = 0$$

$$\boxed{3y + 3(z+1) = 0}$$

4.2 (b) **10 / 10**

✓ **- 0 pts** Correct

- 2 pts wrong normal vector

- 1 pts calculation error

- 10 pts no answer

5. (20 points) Determine if there is a plane that contains the following two lines. If yes, then find the equation of the plane.

$$\mathbf{r}_1(t) = \langle t+2, -2, -t \rangle,$$
$$\mathbf{r}_2(t) = \langle 0, -2t, t+1 \rangle.$$

* a plane can be determined by two lines that intersect at a single point

$x = t_1 + 2 = 0 \Longrightarrow t_1 = -2$ check: $-(-2) = 1+1$

$y = -2 = -2t_2 \Longrightarrow t_2 = 1$ $2 = 2$ ✓

$z = -t_1 = t_2 + 1$

$r_1(-2) = \langle -2+2, -2, -(-2) \rangle = \langle 0, -2, 2 \rangle$

intersection pt: $(0, -2, 2)$

because these two lines intersect at a single point, yes, there is a plane containing these two lines

$\overrightarrow{PA} = \langle 2-0, -2-(-2), 0-2 \rangle = \langle 2, 0, -2 \rangle$

$\overrightarrow{PB} = \langle 0-0, 0-(-2), 1-2 \rangle = \langle 0, 2, -1 \rangle$

$P = \langle 0, -2, 2 \rangle$

$A = r_1(0) = \langle 0+2, -2, -0 \rangle = \langle 2, -2, 0 \rangle$

$B = r_2(0) = \langle 0, -2(0), 0+1 \rangle = \langle 0, 0, 1 \rangle$

$$n = \overrightarrow{PA} \times \overrightarrow{PB} = \begin{vmatrix} i & j & k \\ 2 & 0 & -2 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2 & -1 \end{vmatrix} i - \begin{vmatrix} 2 & -2 \\ 0 & -1 \end{vmatrix} j + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} k$$

$$= \langle 0-(-4), -(-2-0), 4-0 \rangle = \langle 4, 2, 4 \rangle$$

$$4(x-0) + 2(y-(-2)) + 4(z-2) = 0$$

$$\boxed{4x + 2(y+2) + 4(z-2) = 0}$$

5 Problem 5 **20 / 20**

- ✓ **+ 20 pts** **Correct**
- **+ 10 pts** Intersection point
- **+ 7 pts** Normal vector
- **+ 3 pts** Partial points
- **- 4 pts** Wrong direction vectors to compute a normal vector
- **- 2 pts** Mistake
- **+ 0 pts** No points