Math 32A, Lecture 1
Multivariable Calculus

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

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Section: 1A Andrew Krieger 9:00A.m.-9:50A.m Tuesdays

Question	Points	Score
1	10	8
2	10	10
3	10	8
4	10	10
5	10	10
Total:	50	46

Problem 1.

Consider the space curve $\mathbf{r}(s) = \langle \frac{1}{3}(1+s)^{\frac{3}{2}}, \frac{1}{3}(1-s)^{\frac{3}{2}}, \frac{s}{\sqrt{2}} \rangle$.

(a) [3pts.] Show that $\mathbf{r}(s)$ is a unit speed curve.

(b) [4pts.] Find the curvature of $\mathbf{r}(s)$ at $s = 0$, and the unit normal vector $\mathbf{N}$ to the curve at this point. [Hint: In light of part (a), there is a fast way to do this.]

(c) [3pts.] Find the osculating plane to $\mathbf{r}(s)$ at $s = 0$, and say what it represents geometrically.

a) $\vec{r}'(s) = \langle \frac{1}{3}\left(\frac{3}{2}\right)(1+s)^{\frac{1}{2}}, -\frac{1}{3}\left(\frac{3}{2}\right)(1-s)^{\frac{1}{2}}, \frac{1}{\sqrt{2}} \rangle$

3/3

$\|\vec{r}'(s)\| = \sqrt{\frac{1}{4}(1+s) + \frac{1}{4}(1-s) + \frac{1}{2}} = \sqrt{\frac{1}{2}+s-s\,\frac{1}{2}} = \boxed{1}$

b) $\vec{r}'(s) = \langle \frac{1}{2}(1+s)^{\frac{1}{2}}, -\frac{1}{2}(1+s)^{\frac{1}{2}}, \frac{\sqrt{2}}{2} \rangle$

$\vec{r}'(0) = \langle \frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2} \rangle$

$\vec{T}(0) = \langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}}} = 1$

$\vec{T}(0) = \langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$

4/4

$\vec{T}'(s) = \langle \frac{1}{4\sqrt{1+s}}, \frac{+1}{4\sqrt{1+s}}, 0 \rangle$

↖ Diff'n error

$\vec{T}'(0) = \langle \frac{1}{4}, -\frac{1}{4}, 0 \rangle$

$\vec{N}(0) \langle \frac{1}{4}, \frac{+1}{4}, 0 \rangle \frac{1}{\sqrt{\frac{1}{16}+\frac{1}{16}}} = \frac{1}{\sqrt{\frac{1}{8}}}$

$\sqrt{8} = 2\sqrt{2}$

$\vec{N}(0) = \langle \frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2}, 0 \rangle$

$\vec{r}''(s) = \frac{1}{4\sqrt{s+1}}, \frac{-1}{4\sqrt{s+1}}, 0$

$\vec{r}'(0) = \frac{1}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}}$

$\vec{r}''(s) = \langle \frac{1}{4}, -\frac{1}{4}, 0 \rangle$

$K(s) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$

$\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{4} & \frac{+1}{4} & 0 \end{vmatrix}$

$= \frac{1}{4\sqrt{2}} + \left(\frac{1}{4\sqrt{2}}\right) + 0$

$\frac{\frac{1}{16(2)}}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}}} = \boxed{\frac{1}{32}} = K(0)$

c) Osculating plane is the closest plane the space curve is lying in at a particular point. 1/3

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Problem 2.

Recall that one parametrization of the cycloid, the path traced by a point on the edge of a wheel of radius one as the wheel rolls forward, is $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$.

(a) [5pts.] Find the arclength of $\mathbf{r}(t)$ along the interval $0 \le t \le 2\pi$, that is, as the wheel rolls through one full circle. [You may find it helpful to recall the following half-angle identity: $1 - \cos t = 2\sin^2(\frac{t}{2})$.]

(b) [5pts.] At what times t the curve is the point on the edge of the wheel whose motion is parametrized by this cycloid moving with speed 1?

$S = \int_0^{2\pi} \sqrt{4\sin^4\frac{t}{2} + \sin^2 t}\, dt$

$\vec{r}\,'(t) = \langle 1 - \cos t, \sin t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(1-\cos t)^2 + \sin^2 t}$

$= \sqrt{\left(2\sin^2 \frac{t}{2}\right)^2 + \sin^2 t}$

$= \sqrt{4\sin^4\left(\frac{t}{2}\right) + \sin^2 t}$

$u = \frac{t}{2}$

$2\,du = dt$

$S = 2\int_0^{2\pi} \sqrt{4\sin^4 u + \sin^2(2u)}\, du$

$S = 8\int_0^{2\pi} \sqrt{\sin^4 u + \frac{\sin^2 2u}{4}}\, du$

$S = 8\int_0^{2\pi} \sqrt{\sin^4 u + \cancel{\frac{4\sin^2 u\cos^2 u}{4}}}\, du$

$S = 8\int_0^{2\pi} \sqrt{\sin^2 u(\sin^2 u + \cos^2 u)}\, du$

$S = 8\int_0^{2\pi} \sin u\, du$

$S = 8 - \cos \frac{t}{2}\Big|_0^{2\pi} = \boxed{8}$

b) $\sqrt{(1-\cos t)^2 + \sin^2 t} = 1$

$(1 - \cos t)^2 + \sin^2 t = 1$

$1 - 2\cos t + \cos^2 t + \sin^2 t = 1$

$1 + 1 - 2\cos t = 1$

$2 - 2\cos t = 1$

$-2\cos t = -1$

$\cos t = \frac{1}{2}$

$t = \frac{\pi}{3}, \frac{5\pi}{3}$

Problem 3.

Draw the following.

(a) [5pts.] The quadric surface $9x^2 - 4y^2 + z^2 = -36$.

(b) [5pts.] The domain of the function $g(x, y, z) = \sqrt{16 - x^2 - 4y^2 - z^2}$.

a) $\dfrac{9x^2 - 4y^2 + z^2 = -36}{-36}$

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$-\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{36} = 1$

$-\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 - \left(\dfrac{z}{6}\right)^2 = 1$

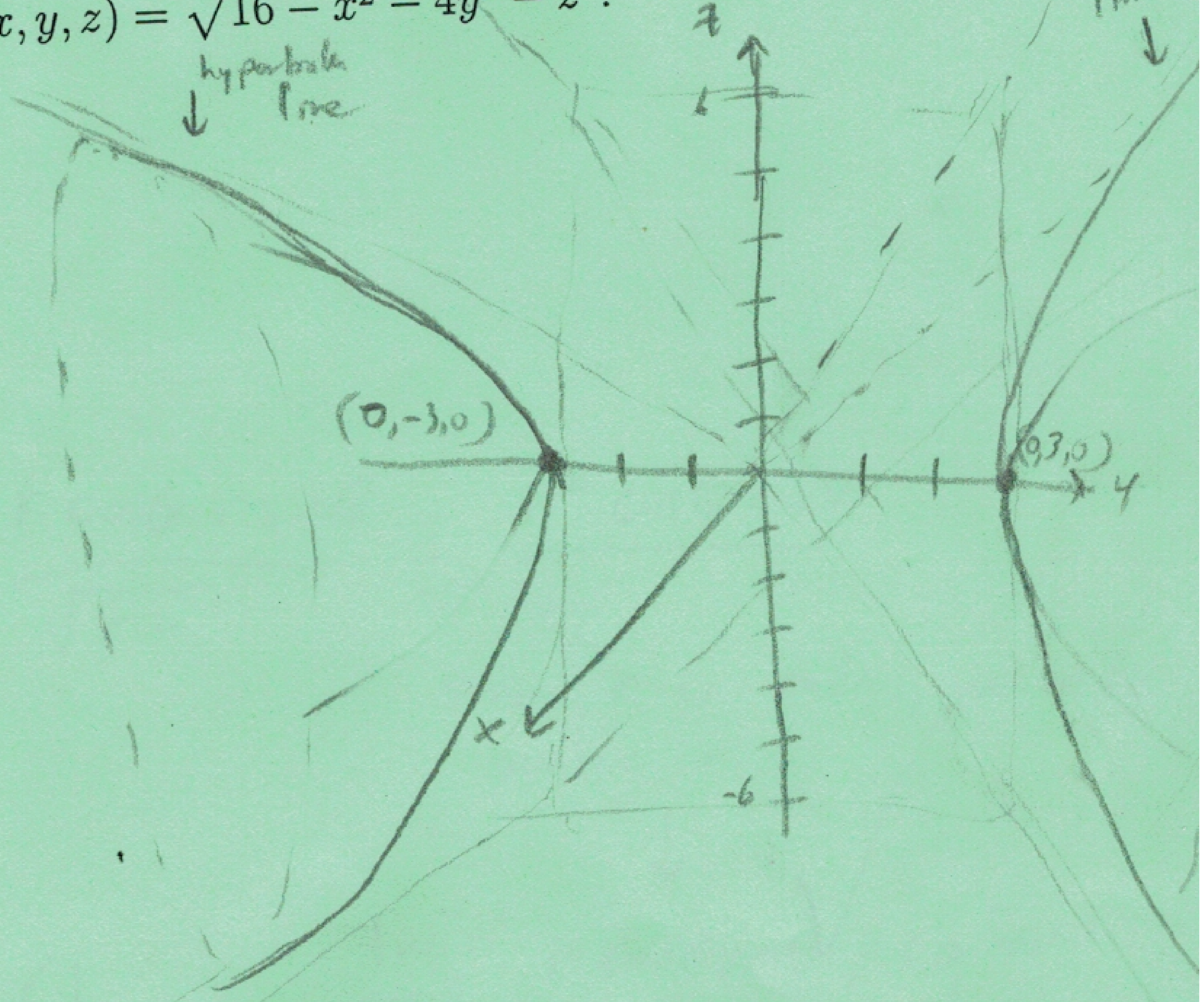
hyperbolad two sheets

$x = 0$

$\left(\dfrac{y}{3}\right)^2 - \left(\dfrac{z}{6}\right)^2 = 1 \rightarrow$ hyperbola

$y = 3$ vertices

$-\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{3}{3}\right)^2 - \left(\dfrac{z}{6}\right)^2 = 1$



b) $g(x,y,z) = \sqrt{16 - x^2 - 4y^2 - z^2}$

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$0^2 \leq 16 - x^2 - 4y^2 - z^2$

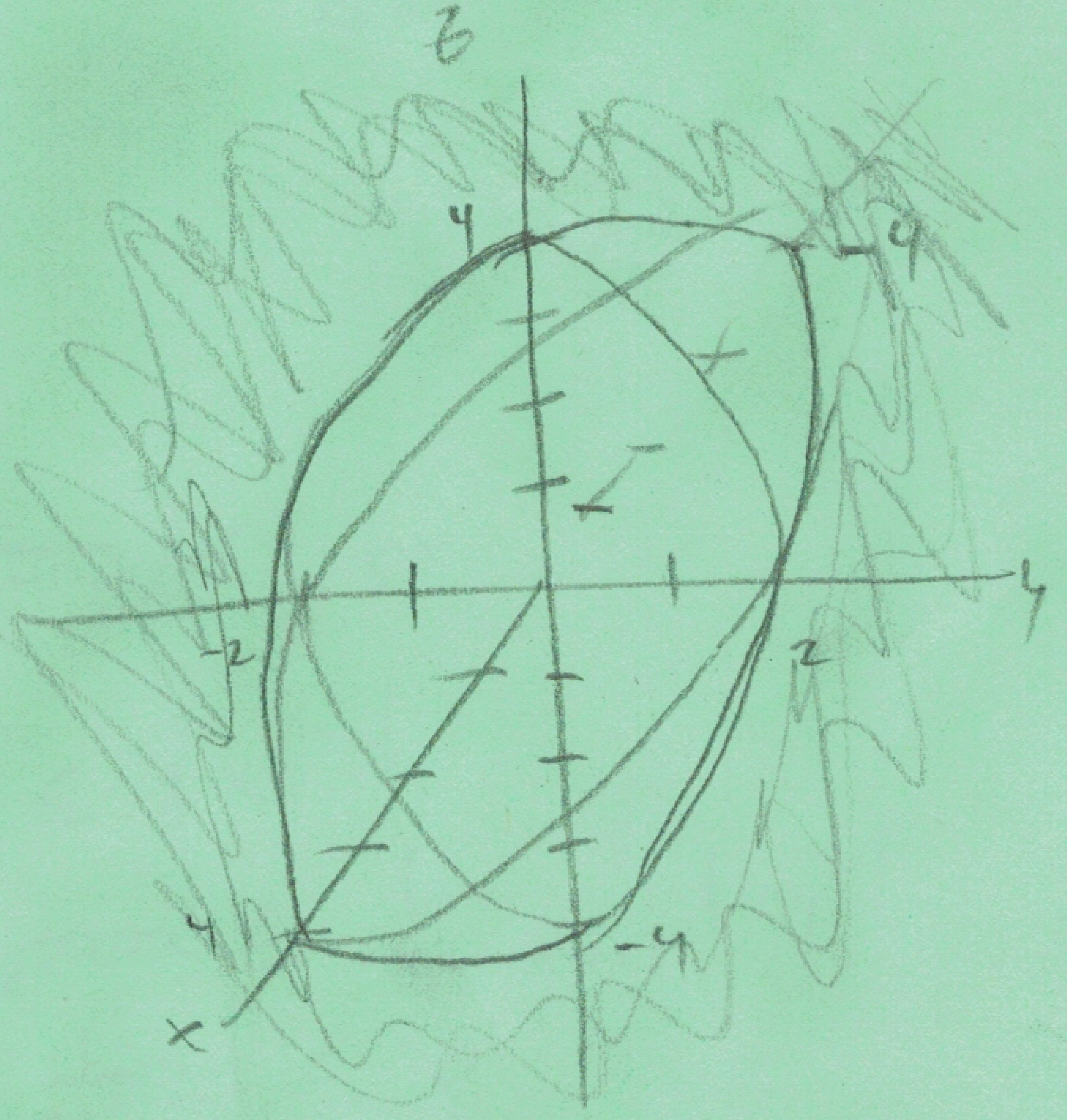
$-16 \geq - x^2 - 4y^2 - z^2$

$\dfrac{x^2}{16} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$

$\left(\dfrac{x}{4}\right)^2 + \left(\dfrac{y}{2}\right)^2 + \left(\dfrac{z}{4}\right)^2 = 1$

Domain is all numbers

x, y, z outside of ellipsoid



$16 - x^2 - 4y^2 - z^2 \geq 0$

Problem 4.

For each of the limits below, either compute the limit or prove that it does not exist. Justify your answers carefully.

(a) [3pts.] $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

(b) [3pts.] $\lim_{(x,y)\to(0,0)} y\tan^{-1}(\frac{1}{x^2})$.

(c) [4pts.] $\lim_{(x,y)\to(0,0)} xy^2\cos(e^{\frac{1}{x^2+50y^2}})$.

a) $x = r\cos\theta \quad y = r\sin\theta$

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$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta\, r\sin\theta}{\sqrt{r^2}} = \frac{r^2\cos\theta\sin\theta}{r}$$

Since r is radius, we take positive r

$\downarrow$

$\lim_{r\to 0} r\cos\theta\sin\theta$, continuous at $r=0$

$\lim_{r\to 0} r\cos\theta\sin\theta = \boxed{0}$

3 b) $\lim_{(x,y)\to(0,0)} y\tan^{-1}\left(\frac{1}{x^2}\right)$

$$= \lim_{(x,y)\to(0,0)} y \cdot \lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{1}{x^2}\right)$$

0 ~~approaches~~ = $\frac{\pi}{2}$

$0\cdot\frac{\pi}{2} = \boxed{0}$

4 c) $\lim_{(x,y)\to(0,0)} xy^2\cos\left(\frac{1}{e^{x^2+50y^2}}\right)$

Note $-1 \le \cos\left(\frac{1}{e^{x^2+50y^2}}\right) \le 1$, so

$$-|xy^2| \le \left|xy^2\cos\left(\frac{1}{e^{x^2+50y^2}}\right)\right| \le |xy^2|$$

$$-\lim_{(x,y)\to(0,0)} |xy^2| \le \lim_{(x,y)\to(0,0)} xy\cos\frac{1}{e^{x^2+50y^2}} \le \lim_{(x,y)\to(0,0)} |xy^2|$$

$$0 \le \lim_{(x,y)\to(0,0)} xy\cos\frac{1}{e^{x^2+50y^2}} \le 0$$

By Squeeze Theorem, $\lim_{(x,y)\to(0,0)} xy^2\cos\frac{1}{e^{(x^2+50y^2)}} = \boxed{0}$

Problem 5.

(a) [5pts.] Compute the first partial derivatives of $g(x, y, z) = xye^{z-x}$.

(b) [5pts.] Either give an example of a function $f(x, y)$ with partial derivatives $f_x(x, y) = 3y^2 - \sin x$ and $f_y(x, y) = 6y + \cos x$, or explain why one cannot exist.

a) $g(x,y,z) = xye^{z-x}$

$g_x(x,y,z) = [(1)ye^{z-x}] + -yxe^{z-x} = \boxed{ye^{z-x}(1-x)}$

$g_y(x,y,z) \boxed{xe^{z-x}}$ $\boxed{g_z = xye^{z-x}}$

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b) $3y^2x + \cos(x)$ has $f_x(x,y) = 3y^2 - \sin x$,

but $3y^2 + (\cos x)y$ would have $f_y(x,y)$

$3y^2 + \cos x y \neq 3y^2 - \sin x$

Such a function does not exist

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$f_{xy} = 6y$ $f_{yx} = \sin x$,

Also by Clairaut's theorem, such functions should have the same higher order derivatives, $f_{xy} = f_{yx}$; however $6y \neq \sin x$.