21F-MATH32A-1 Midterm 2

[redacted]

TOTAL POINTS

100 / 100

QUESTION 1

1 Question 1 25 / 25

✓ **- 0 pts** Correct

- 12 pts (ii)is not solved or incorrect

- 5 pts Algebra mistake in (ii) OR in (i) and (ii)

- 25 pts Did not solve the problem at all or solve the two questions incorrectly

QUESTION 2

2 Question 2 25 / 25

✓ **- 0 pts** Correct.

- 2 pts Does not conclude $$f$$ is continuous at all points in part (i).

- 10 pts Fails to prove $$f$$ is continuous at $$(0,0)$$ in part (i) using Squeeze Theorem.

- 3 pts Fails to conclude limit equals 0 in part (ii).

- 10 pts Fails to prove limit exists in part (ii) using Squeeze Theorem.

- 0 pts Correct argument with minor errors (wrong inequality, missing absolute value where needed, etc).

+ 3 pts Partial credit for (i) - correctly computed the limit along some straight line paths.

+ 6 pts Partial credit for (i) - used polar coordinates to correctly compute the limit in a nonrigorous way.

+ 3 pts Partial credit for (ii) - correctly computed the limit along some straight line paths.

+ 6 pts Partial credit for (ii) - used polar coordinates to correctly compute the limit in a nonrigorous way.

- 0 pts Click here to replace this description.

QUESTION 3

3 Question 3 25 / 25

✓ **+ 10 pts** Took two derivatives to obtain acceleration.

✓ **+ 5 pts** Found unit tangent vector (or equivalent information).

✓ **+ 5 pts** Used unit tangent vector (or equivalent information) to find tangential and normal components.

✓ **+ 5 pts** Correctly evaluated at $$t=\frac{\pi}{2}.$$

+ 0 pts Click here to replace this description.

QUESTION 4

4 Question 4 25 / 25

✓ **+ 5 pts** Correct expression for $$f_t$$ with work

✓ **+ 5 pts** Correct expression for $$f_{xx}$$ with work

✓ **+ 5 pts** Correct expression for $$f_{yy}$$ with work

✓ **+ 5 pts** Correct differential equation for $$h(t)$$ with work

✓ **+ 5 pts** Correct expression for $$h(t)$$

+ 0 pts Blank or incorrect

+ 5 pts (Partial credit) Calculated first derivatives correctly

- 2 pts Small error

Midterm 02

Math 32a @ $\mathscr{U}CLA$ (Fall 2021)

[redacted]

Instructions/Admonishment

1. SHOW ALL WORK.
   A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. You may not use books, calculators, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it.
2. Duration: 50 min.
3. The following is my own work, without the aid of any other person.
   Signature: [redacted]

Please do not write below this line.

Problem	Points	Scores
1	25	
2	25	
3	25	
4	25	
Total	100	

Question 1 ***The curvature.***

Consider the curve with the vector equation $\mathbf{r}(t) = \langle 3\sin(t), 4t, -3\cos(t)\rangle$.

(i) For what value(s) of t does $\mathbf{r}(t)$ lie on the sphere $x^2 + y^2 + z^2 = 57$?

(ii) Find the unit tangent vector $\mathbf{T}(t)$ and the curvature $\kappa(t)$.

i. when r(t) lies on the sphere, $x = 3\sin t$, $y = 4t$, $z = -3\cos t$

$(3\sin t)^2 + (4t)^2 + (-3\cos t)^2 = 57$

$9\sin^2 t + 16t^2 + 9\cos^2 t = 57$

$16t^2 + 9 = 57$

$16t^2 = 48$

$\boxed{t = \pm\sqrt{3}}$

ii.

$T(t) = \frac{r'(t)}{\|r'(t)\|}$

$r'(t) = \langle 3\cos t, 4, 3\sin t\rangle$

$\|r'(t)\| = \sqrt{(3\cos t)^2 + 4^2 + (3\sin t)^2} = \sqrt{9\cos^2 t + 4^2 + 9\sin^2 t} = \sqrt{9+16} = \sqrt{25} = 5$

so $T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 3\cos t, 4, 3\sin t\rangle}{5} = \boxed{\langle \frac{3}{5}\cos t, \frac{4}{5}, \frac{3}{5}\sin t\rangle}$

$\kappa(t) = \frac{1}{\|r'(t)\|} \cdot \left\|\frac{d}{dt} T(t)\right\|$

$\frac{d}{dt} T(t) = \langle -\frac{3}{5}\sin t, 0, \frac{3}{5}\cos t\rangle$

$\left\|\frac{d}{dt} T(t)\right\| = \sqrt{(-\frac{3}{5}\sin t)^2 + 0 + (\frac{3}{5}\cos t)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\kappa(t) = \frac{1}{5} \cdot \frac{3}{5} = \boxed{\frac{3}{25}}$

Question 2 ***The limit and continuity.***

(i) At what points is $f(x, y)$ continuous?

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}; & (x,y) \neq (0,0). \\ 0; & (x,y) = (0,0). \end{cases}$$

(ii) Find $\lim_{(x,y)\to(0,0)} \left(\dfrac{xy}{\sqrt{x^2+y^2}} \right)$.

i. x^3+y^3 is continuous on $\mathbb{R}\times\mathbb{R}$

$\frac{1}{x^2+y^2}$ is continuous on $\mathbb{R}\times\mathbb{R} \setminus \{(0,0)\}$

so $\frac{x^3+y^3}{x^2+y^2}$ is continuous on $\mathbb{R}\times\mathbb{R} \setminus \{(0,0)\}$

at (x,y) of $(0,0)$, $f(x,y)=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \frac{0+0}{0+0} = \frac{0}{0}$$

↓ switch to polar

$x = r\cos\theta$

$y = r\sin\theta$

$$\lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = r(\cos^3\theta+\sin^3\theta)$$

$-1 \le \cos\theta \le 1$ $\quad$ $-1 \le \sin\theta \le 1$

$|\cos\theta| \le 1$ $\quad$ $|\sin\theta| \le 1$

$|\cos^3\theta| \le 1$ $\quad$ $|\sin^3\theta| \le 1$

$|\cos^3\theta| + |\sin^3\theta| \le 2$

$|\cos^3\theta + \sin^3\theta| \le |\cos^3\theta| + |\sin^3\theta|$

so

$0 \le |\cos^3\theta + \sin^3\theta| \le 2$

$0 \le |r(\cos^3\theta + \sin^3\theta)| \le 2|r|$

$\lim_{r\to 0} 0 = 0$; $\lim_{r\to 0} 2|r| = 0$,

so by squeeze theorem,

$\lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$ → $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = f(0,0) = 0$

$\therefore f(x,y)$ is continuous on $(0,0)$

so $f(x,y)$ is continuous for all $\mathbb{R}\times\mathbb{R}$

ii.

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \frac{0\cdot 0}{\sqrt{0+0}} = \frac{0}{0}$$

↓

switch to polar

$x = r\cos\theta$

$y = r\sin\theta$

$$\lim_{r\to 0} \left(\frac{r^2\cos\theta\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}\right) = \lim_{r\to 0}\left(\frac{r^{\not 2}\cos\theta\sin\theta}{\not r}\right) = \lim_{r\to 0} r\cos\theta\sin\theta = \lim_{r\to 0} \frac{r\sin 2\theta}{2}$$

$$-1 \le \sin 2\theta \le 1$$

$$0 \le |\sin 2\theta| \le 1$$

$$0 \le \frac{|\sin 2\theta|}{2} \le \frac{1}{2}$$

$$0 \le \left|\frac{r\sin 2\theta}{2}\right| \le \frac{1}{2}|r|$$

$\lim_{r\to 0} 0 = 0$ and $\lim_{r\to 0} \frac{1}{2}|r| = 0$

$\therefore \lim_{r\to 0} \frac{r\sin 2\theta}{2} = 0$ by the squeeze theorem

↓

$$\boxed{\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right) = 0}$$

Question 3 *The acceleration.*
Suppose $\mathbf{r}(t) = \langle \sin(t), \cos(t), t \rangle$.

(i) Find the acceleration vector $\mathbf{a}(t)$.

(ii) At $t = \frac{\pi}{2}$ decompose the acceleration vector into its tangential and normal components.

i. $r(t) = \langle \sin t, \cos t, t \rangle$

$r'(t) = v(t) = \langle \cos t, -\sin t, 1 \rangle$

$r''(t) = \boxed{a(t) = \langle -\sin t, -\cos t, 0 \rangle}$

ii. $T = \frac{v(t)}{\|v(t)\|} = \frac{\langle \cos t, -\sin t, 1 \rangle}{\sqrt{\cos^2 t + \sin^2 t + 1}} = \langle \frac{\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

since $\|v(t)\| = \sqrt{2}$ for all t (constant speed), there is no tangential acceleration.

$a_T = 0$

$a_N = \sqrt{\|a\|^2 - |a_T|^2} = \sqrt{\|a\|^2} = \|a\| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + 0^2} = \sqrt{1} = 1$

$a_N N = a - a_T T = a = \langle -\sin t, -\cos t, 0 \rangle$

$N = \frac{a_N N}{a_N} = \frac{\langle -\sin t, -\cos t, 0 \rangle}{1} = \langle -\sin t, \cos t, 0 \rangle$

so $a = N$

at $t = \frac{\pi}{2}$, $N = \langle -\sin\frac{\pi}{2}, -\cos\frac{\pi}{2}, 0 \rangle = \langle -1, 0, 0 \rangle$

$\boxed{\text{so } a = N. \ (a = 0T + 1N), \text{ where } N = \langle -1, 0, 0 \rangle}$

Question 4

Consider the function defined by $f(t, x, y) = e^{h(t)} \sin(x) \cos(y)$.

Find the function $h(t)$ such that f is a solution to the equation $f_t = f_{xx} + f_{yy}$.

$f(t,x,y) = e^{h(t)} \sin x \cos y$

$f_t = h'(t) e^{h(t)} \sin x \cos y$

$f_x = e^{h(t)} \cos x \cos y$ $\quad$ $f_y = -e^{h(t)} \sin x \sin y$

$f_{xx} = -e^{h(t)} \sin x \cos y$ $\quad$ $f_{yy} = -e^{h(t)} \sin x \cos y$

$f_t = f_{xx} + f_{yy}$

$h'(t) e^{h(t)} \sin x \cos y = -e^{h(t)} \sin x \cos y + (-e^{h(t)} \sin x \cos y)$

$h'(t) e^{h(t)} \sin x \cos y = -2 e^{h(t)} \sin x \cos y$

$h'(t) = -2$

$h(t) = \int h'(t)\,dt = \int -2\,dt = -2t + C$

so $\boxed{h(t) = -2t + C}$