Midterm 02

Math 32a @ $\mathscr{U}CLA$ (Fall 2021)

Assigned: November 19, 2021.

Last Name: ______________________

First Name: ______________________

Student ID: ______________________

Instructions/Admonishment

1. SHOW ALL WORK.
   A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. You may not use books, calculators, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it.
2. Duration: 50 min.
3. The following is mv own work, without the aid of any other person.
   Signature: ______________________

Please do not write below this line.

Problem	Points	Scores
1	25	
2	25	
3	25	
4	25	
Total	100	

Question 1 ***The curvature.***
Consider the curve with the vector equation $\mathbf{r}(t) = \langle 3\sin(t), 4t, -3\cos(t)\rangle$.

(i) For what value(s) of t does $\mathbf{r}(t)$ lie on the sphere $x^2 + y^2 + z^2 = 57$?

(ii) Find the unit tangent vector $\mathbf{T}(t)$ and the curvature $\kappa(t)$.

(i) $r(t) = \langle 3\sin(t), 4t, -3\cos(t)\rangle$

$x(t) = 3\sin t \quad y(t) = 4t \quad z(t) = -3\cos t$

$(3\sin t)^2 + (4t)^2 + (-3\cos t)^2 = 57$

$9\sin^2 t + 16t^2 + 9\cos^2 t = 57$

$9(\sin^2 t + \cos^2 t) + 16t^2 = 57$

$9 + 16t^2 = 57$

$16t^2 = 48$

$\sqrt{t^2} = \sqrt{3}$

$\boxed{t = \pm\sqrt{3}}$

(ii) $r(t) = \langle 3\sin t, 4t, -3\cos t\rangle$

$r'(t) = \langle 3\cos t, 4, 3\sin t\rangle$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 3\cos t, 4, 3\sin t\rangle}{\sqrt{(3\cos t)^2 + (4)^2 + (3\sin t)^2}} = \frac{\langle 3\cos t, 4, 3\sin t\rangle}{\sqrt{9(\cos^2 t + \sin^2 t) + 16}}$$

$$= \frac{\langle 3\cos t, 4, 3\sin t\rangle}{5}$$

$$\boxed{T(t) = \left\langle \frac{3}{5}\cos t, \frac{4}{5}, \frac{3}{5}\sin t\right\rangle}$$

$\kappa(t) = \frac{1}{\|r'(t)\|}\left\|\frac{dT}{dt}\right\|$

$T'(t) = \left\langle -\frac{3}{5}\sin t, 0, \frac{3}{5}\cos t\right\rangle$

$\kappa(t) = \frac{1}{5}\left(\sqrt{(-\frac{3}{5}\sin t)^2 + 0^2 + (\frac{3}{5}\cos t)^2}\right)$

$\kappa(t) = \frac{1}{5}\left(\sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}\right)$

$\kappa(t) = \frac{1}{5}\cdot\frac{3}{5} = \boxed{\frac{3}{25}}$

Question 2 *The limit and continuity.*

(i) At what points is $f(x, y)$ continuous?

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}; & (x,y) \neq (0,0). \\ 0; \ (x,y) = (0,0). \end{cases}$$

(ii) Find $\lim_{(x,y)\to(0,0)} \left(\dfrac{xy}{\sqrt{x^2+y^2}} \right)$.

(i) x^3+y^3 is a polynomial function and thus continuous on $\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$, therefore it is continuous on $\mathbb{R}\times\mathbb{R}\setminus\{(0,0)\}$

$\frac{1}{x^2+y^2}$ is a rational function and continuous on its domain $\mathbb{R}\times\mathbb{R}\setminus\{(0,0)\}$

$f(x,y)=\frac{x^3+y^3}{x^2+y^2}$ is continuous on the domain $\mathbb{R}\times\mathbb{R}\setminus\{(0,0)\}$ because it is the ratio of two functions continuous on $\mathbb{R}\times\mathbb{R}\setminus\{(0,0)\}$

at $f(0,0)$:

$\boxed{f(0,0)=0}$

$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

$x=r\cos\theta \quad y=r\sin\theta$

$\lim_{r\to0} \frac{r^3\cos^3\theta+r^3\sin^3\theta}{r^2\cos^2\theta+r^2\sin^2\theta}$

$\lim_{r\to0} \frac{r^{\cancel{3}}(\cos^3\theta+\sin^3\theta)}{\cancel{r^2}}$

$\lim_{r\to0} r(\cos^3\theta+\sin^3\theta)$

$\lim_{(x,y)\to(0,0)} \left|\frac{x^3+y^3}{x^2+y^2}\right| = \lim_{r\to0} |r(\cos^3\theta+\sin^3\theta)|$

$|\cos\theta|\le1 \Rightarrow |\cos^3\theta|\le1$

$|\sin\theta|\le1 \Rightarrow |\sin^3\theta|\le1$

$|\cos^3\theta+\sin^3\theta|\le|\cos^3\theta|+|\sin^3\theta|$

$|\cos^3\theta+\sin^3\theta|\le1+1=2$

$0\le \lim_{(x,y)\to(0,0)} \left|\frac{x^3+y^3}{x^2+y^2}\right| = \lim_{r\to0} |r(\cos^3\theta+\sin^3\theta)| \le 2r$

$\lim_{r\to0} 0 \le \lim_{(x,y)\to(0,0)} \left|\frac{x^3+y^3}{x^2+y^2}\right| = \lim_{r\to0}|r(\cos^3\theta+\sin^3\theta)| \le \lim_{r\to0} 2r = 0 \Rightarrow$

therefore by the squeeze thm, $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \boxed{0}$

Because $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = f(0,0)$ the function $f(x,y)$ is continuous at $(0,0)$.

therefore, $f(x,y)$ is continuous on $\mathbb{R}\times\mathbb{R}$.

ii) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

$x=r\cos\theta \quad y=r\sin\theta$

$\lim_{r\to0} \frac{r\cos\theta(r\sin\theta)}{\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}}$

$\lim_{r\to0} \frac{r^{\cancel{2}}\cos\theta\sin\theta}{\cancel{r}}$ $\quad \lim_{r\to0} r\cos\theta\sin\theta$

$|\cos\theta|\le|\sin\theta|\le1$ $\quad 0\le\lim_{r\to0}|r\cos\theta\sin\theta|\le1\cdot r = r$

$|\cos\theta(\sin\theta)|\le1\cdot1$

$|\cos\theta\sin\theta|\le1$

$0\le\lim_{(x,y)\to(0,0)}\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \lim_{r\to0}|r\cos\theta\sin\theta| \le \lim_{r\to0} r$

$0\le\lim_{(x,y)\to(0,0)}\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \lim_{r\to0}|r\cos\theta\sin\theta|\le0$

By the squeeze theorem, $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

Question 3 ***The acceleration.***

Suppose $\mathbf{r}(t) = \langle \sin(t), \cos(t), t \rangle$.

(i) Find the acceleration vector $\mathbf{a}(t)$.

(ii) At $t = \frac{\pi}{2}$ decompose the acceleration vector into its tangential and normal components.

(i) $r(t) = \langle \sin t, \cos t, t \rangle$

$v(t) = r'(t) = \langle \cos t, -\sin t, 1 \rangle$

$$\boxed{a(t) = r''(t) = \langle -\sin t, -\cos t, 0 \rangle}$$

(ii) $t = \frac{\pi}{2}$

$$T = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle \cos t, -\sin t, 1 \rangle}{\sqrt{\cos^2 t + (-\sin t)^2 + 1^2}} = \frac{1}{\sqrt{2}} \langle \cos t, -\sin t, 1 \rangle$$

$T(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle$ $\quad a(\frac{\pi}{2}) = \langle -1, 0, 0 \rangle$

$$a_T = a \cdot T = \langle -1, 0, 0 \rangle \cdot \langle 0, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle$$
$$= 0 + 0 + 0 = 0$$

$\underline{a_T = 0}$

therefore

$$\vec{a} = \underbrace{a_T}_{0} \vec{T} + a_N \vec{N}$$

$$\vec{a} = a_N \vec{N}$$

$$\langle -1, 0, 0 \rangle = a_N \vec{N}$$

$$a_N = \|a_N \vec{N}\| = \sqrt{(-1)^2 + 0^2 + 0^2} = \underline{1}$$

$$\vec{N} = \frac{a_N \vec{N}}{a_N} = \frac{\langle -1, 0, 0 \rangle}{1} = \underline{\langle -1, 0, 0 \rangle}$$

$\underline{\vec{a} = 0T + 1\vec{N}}$

$\vec{a} = 0\vec{T} + 1\vec{N} \Rightarrow \vec{a} = 1\vec{N}$

$a_T = 0 \quad a_N = 1$

$\vec{N} = \langle -1, 0, 0 \rangle$

$\vec{T} = \langle 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\|v\| = \|r'(t)\| = \sqrt{\cos^2 t + (-\sin^2 t) + 1^2}$
$= \sqrt{2}$

speed is constant so a_T will be 0 because there is no tangential component!

Question 4

Consider the function defined by $f(t,x,y) = e^{h(t)}\sin(x)\cos(y)$.

Find the function $h(t)$ such that f is a solution to the equation $f_t = f_{xx} + f_{yy}$.

$f(t,x,y) = e^{h(t)} \sin(x)\cos(y)$

$f_t = h'(t)\, e^{h(t)} \sin x \cos y$

$f_x = e^{h(t)} \cos x \cos y$

$f_{xx} = -\sin x\, e^{h(t)} \cos y = -e^{h(t)} \sin x \cos y$

$f_y = e^{h(t)} \sin x(-\sin y)$

$f_{yy} = -e^{h(t)} \sin x \cos y$

$f_t = f_{xx} + f_{yy}$

$h'(t)\, e^{h(t)} \sin x \cos y = - e^{h(t)} \sin x \cos y - e^{h(t)} \sin x \cos y$

$h'(t)\, e^{h(t)} \sin x \cos y = -2e^{h(t)} \sin x \cos y$

$h'(t) = -2$

$\int h'(t)\,dt = \int -2\,dt$

$\boxed{h(t) = -2t + C}$