TEST 2

MATH 32A @ UCLA (FALL 2020)

Assigned: December 02, 2020.

Instructions/Admonishment

1. SHOW ALL WORK.
   A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit.
2. Duration: 24 hours.
3. The following is my own work, without the aid of any other person.
   Signature: ____

Exercise 1 TRUE OR FALSE.

State whether the folowing statements are $TRUE$ or $FALSE$. Explain your choice.

(i) Every element of the range of $f(x, y)$ is contained in some level curve of $f(x, y)$.

(ii) There is a function $f(x, y)$ with continuous second partial derivative such that $f_x = f_y = e^x \sin(xy)$.

Exercise 2 EXAMPLES.

Give one example of:

(i) A function $f(x, y)$ for which *Clairaut′s Theorem* of equality of mixed partials does not apply. Explain.

(ii) A function $f(x, y)$ with domain and range, $\{(x, y) \,|\, x > 0\}$ and $\mathbb{R}$, respectively. Explain.

Exercise 3 LIMIT AND CONTINUITY.

(i) At what points is $f(x,y)$ continuous?

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}\,;\ (x,y) \neq (0,0). \\ 1;\ (x,y) = (0,0). \end{cases}$$

(ii) Find $\lim_{(x,y)\to(0,0)} \left(\dfrac{x^2+y^2}{\sqrt{x^2+y^2+4}-2} \right)$.

Exercise 4 DIFFERENTIABILITY .

Consider the function

$$f(x,y) = \begin{cases} -\dfrac{xy}{x^2+y^2}\,;\ (x,y) \neq (0,0). \\ 0;\ (x,y) = (0,0). \end{cases}$$

(i) Prove that $f(x,y)$ is discontinous at $(0,0)$? Give the reason why $f(x,y)$ is not differentiable at $(0,0)$.

(ii) Use the limit definition of partial derivative to compute $f_x(0,0)$ and $f_y(0,0)$. What can you conclude?

Exercise 5 DIRECTIONAL DERIVATIVE.

Let $f(x,y) = \ln(xy)$.

(i) Find $\nabla f(\frac{1}{2}, \frac{1}{3})$ and the directional derivative of $f(x,y)$ at $(\frac{1}{2}, \frac{1}{3})$ in the direction of the vector $\mathbf{v} = \langle 1, 2 \rangle$.

(ii) Find the direction in which the directional derivative of $f(x,y)$ at $(\frac{1}{2}, \frac{1}{3})$ is largest. Find this largest value.

Exercise 6 CHAIN RULE.

A rectangle has sides of length x and y, which change at the rates $\dfrac{dx}{dt} = 2$ and $\dfrac{dy}{dt} = -1$. If $x = 3$ and $y = 4$, find the rate of change of the length of a diagonal of the rectangle.

Exercise 7 BEAUTIFUL HEAT EQUATION (BONUS QUESTION).

The flow of heat along a thin conducting bar is governed by the one-dimensional heat equation, $\dfrac{\partial T}{\partial t} = k\dfrac{\partial^2 T}{\partial x^2}$, where T is a measure the temperature at a location x of the bar at time t and the positive constant k is related to the conductivity of the material.

Show that the following function satisfy the heat equation with $k = 1$: $T(x,t) = Ae^{-a^2 t}\cos(ax)$ for any real numbers a and A.