~~32B~~ Killip
32A

Midterm 1

Oct 25th

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Section: 1D =

- 1*a* Tuesday with Derek Levinson
- 1*b* Thursday with Derek Levinson
- 1*c* Tuesday with Allen Boozer
- 1*d* Thursday with Allen Boozer
- 1*e* Tuesday with Alan Zhou
- 1*f* Thursday with Alan Zhou

Rules.

- There are **FOUR** problems; ten points per problem.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...
- Use the backs of pages as necessary.

1	2	3	4	Σ
9	6	7	5	27

(1) Consider the three points

$$\vec{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \vec{c} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

(a) What is the area of the triangle with vertices $\vec{a}$, $\vec{b}$, and $\vec{c}$?

(b) What is the equation of the plane passing through these three points?

(c) Give a parametric description of the line passing through $\vec{a}$ that is perpendicular to the plane through the three points.

a) $A_\Delta = \dfrac{\|\vec{v} \times \vec{w}\|}{2}$

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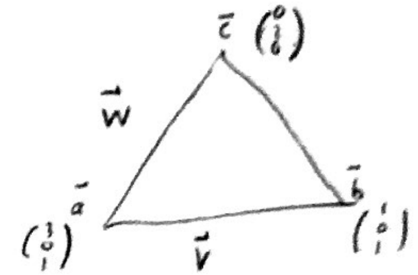
$\vec{v} = \vec{b} - \vec{a} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$

$w = \vec{c} - \vec{a} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$

$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 0 \\ -3 & 2 & -1 \end{vmatrix}$

$= (0-0)\vec{i} - (2-0)\vec{j} + (-4 + 0)\vec{k}$

$= -2\vec{j} - 4\vec{k}$



$\|\vec{v} \times \vec{w}\| = \sqrt{(-2)^2 + (-4)^2} = \sqrt{4 + 16} = 2\sqrt{5}$

Area of triangle $= \dfrac{2\sqrt{5}}{2} = \sqrt{5}$ units ^{2} ✓

b) $P = \vec{n} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = d$

$\vec{n} = \vec{v} \times \vec{w}$ because the Plane P passes through both v and w, so their normal is the binormal vector.

1

$\vec{n} = -2\vec{j} - 4\vec{k}$ (from above)

Point on the plane $= \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$d = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}$

$P = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ ✗

c) $\vec{a} + t\vec{n}$ is a line passing through $\vec{a}$ that is perpendicular to the plane because all $\vec{n}$ is perpendicular to any point on the plane

3

$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}$ ✓

(2) (a) Complete the statement of the Cauchy–Schwarz inequality:

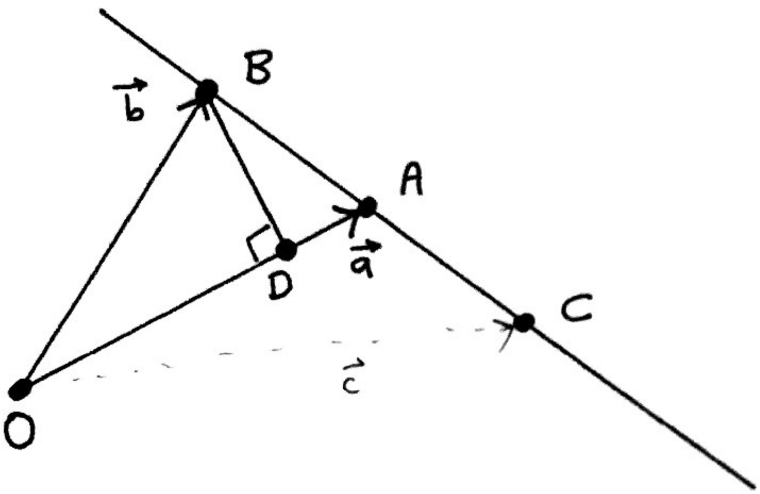
$\frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 b_2^2}}$

$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \cdot \|\vec{b}\|$ ✓

(b) The volume of the parallelepiped spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by...

$|\vec{c} \cdot (\vec{a} \times \vec{b})|$ ✓

Parts (c) and (d) relate to the diagram below. Note that the positions of the points A and B relative to the origin O are given by vectors $\vec{a}$ and $\vec{b}$.



(c) The point C is defined as lying on the line through A and B so that the distance from A to C is the same as that from A to B. Express the location of C in terms of the vectors $\vec{a}$ and $\vec{b}$.

$\vec{BA} = \vec{AC}$ ✓

$c = \vec{a} + \vec{AC}$

$c = \vec{b} + \vec{BC}$

$\vec{BA} = \vec{b} - \vec{a}$

$\vec{BA} = \vec{AC}$

$\vec{BC} = \vec{BA} + \vec{AC}$

$-\vec{AC} = \vec{b} - \vec{a}$

$\vec{BC} = 2(\vec{AC})$ since $\vec{AC} = \vec{BA}$

$\vec{c} = \vec{b} + \vec{BC}$

$= \vec{b} + \cancel{2(\vec{b} - \vec{a})}$ $\cancel{2}(\cancel{a} - \vec{b})$

$= 3\vec{b} - 2\vec{a}$

$\vec{c} = 3\vec{b} - 2\vec{a}$

since $2\vec{a}$ gives the $\vec{BA} + \vec{AC}$ and $\vec{BA}$ and $\vec{AC}$ are of equal length, $\vec{c} = 3\vec{b} - 2\vec{a}$

(d) The point D is defined as lying on the line through O and A and having the property that the line $\overline{BD}$ meets $\overline{OA}$ at right angles. Express the location of D in terms of the vectors $\vec{a}$ and $\vec{b}$.

$\vec{a} \cdot \vec{b} = 0$

$\vec{BD} \cdot \vec{OA} = 0$

(3) The velocity of a particle is given by

$$\vec{v}(t) = \begin{pmatrix} t \\ 3 \\ t^2 \end{pmatrix}$$

(a) Given that the particle is at the point $(1, 2, 0)$ at time $t = 0$, determine its location at time $t = 1$.
(b) Determine the curvature of the path taken by the particle at the point where $t = 0$.
(c) Find the tangential component of the acceleration of the particle at time $t = 1$.

a) $\vec{v}(t) = \begin{pmatrix} t \\ 3 \\ t^2 \end{pmatrix}$

$\int v(t)\,dt$ = position

$\int v(t)\,dt = \begin{pmatrix} \frac{t^2}{2} \\ 3t \\ \frac{t^3}{3} \end{pmatrix} + C$

At $t=0$, position $= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

At $t=1$, position is

$\begin{pmatrix} \frac{1^2}{2} + 1 \\ 3(1) + 2 \\ \frac{1^3}{3} + 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 1 \\ 3 + 2 \\ \frac{1}{3} + 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 5 \\ \frac{1}{3} \end{pmatrix}$ ✓

b) $K(t) = \dfrac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$ good $\quad$ or $\quad \vec{v}\,'(t)$

$\vec{v}(t) = \sqrt{t^2 + 3^2 + t^4} = \sqrt{t^4 + t^2 + 9}$

$\vec{v}\,'(t) = \frac{1}{2}(4t^3 + 2t)(t^4 + t^2 + 9)^{-1/2} = K(t)$

At $t=0$,

$K(t=0) = \frac{1}{2}(4(0)^3 + 2(0))(0^4 + 0^2 + 9)^{-1/2}$

$= \frac{1}{\sqrt{9}} = \frac{1}{3}$ $\quad$ X -2

c) $\vec{a} = a_T \vec{T} + a_N \vec{N}$

$a_T \vec{T} = \dfrac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$

$\vec{a}(t) = \vec{v}\,'(t)$

$= \begin{pmatrix} 1 \\ 0 \\ 2t \end{pmatrix}$ good

$\dfrac{\begin{pmatrix} 1 \\ 0 \\ 2t \end{pmatrix} \cdot \begin{pmatrix} t \\ 3 \\ t^2 \end{pmatrix}}{\begin{pmatrix} t \\ 3 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} t \\ 3 \\ t^2 \end{pmatrix}} \begin{pmatrix} t \\ 3 \\ t^2 \end{pmatrix} =$

$\dfrac{t + 0 + 2t^3}{t^2 + 9 + t^4} \cdot \begin{pmatrix} t \\ 3 \\ t^2 \end{pmatrix}$

At $t = 1$, $a_T \vec{T} =$

$\dfrac{(1) + 0 + 2(1)^3}{(1)^2 + 9 + (1)^4} \cdot \begin{pmatrix} 1 \\ 3 \\ 1^2 \end{pmatrix} =$

$\frac{3}{11} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} =$

$\frac{3}{11} + \frac{9}{11} + \frac{3}{11} = \frac{15}{11}$

$a_T \vec{T}$ at $t=1$ $= \frac{15}{11}$ X $-$

(4) Consider the curve parameterized by

$$\vec{r}(t) = \begin{pmatrix} 2t \\ e^t + e^{-t} \end{pmatrix}. \quad 3+2=5$$

(a) Determine the length of arc between parameter values $t = 0$ and $t = 3$.
(b) Determine $\kappa(t)$, that is, the curvature as a function of the parameter t.

a) $S(t) = \int_0^3 \|\vec{r}\,'(t)\|\,dt$

$(e^t - e^{-t})(e^t - e^{-t})$

$e^{2t} - \cancel{e^{t-t}} - \cancel{e^{t-t}} + e^{-2t}$

$\vec{r}\,'(t) = \begin{pmatrix} 2 \\ e^t - e^{-t} \end{pmatrix}$

$\|\vec{r}\,'(t)\| = \sqrt{2^2 + (e^t - e^{-t})^2} = \sqrt{4 + e^{2t} - 2e^0 + e^{-2t}}$

$= \sqrt{4 + e^{2t} + e^{-2t}}$ should be 2

$e^0 \neq 0$

$= \int_0^3 \sqrt{4 + e^{2t} + e^{-2t}}\,dt$

$e^{2t} + \frac{1}{e^{2t}} + 4$

$u = 2t$

$\frac{e^{4t} + 4e^{2t}}{e^{2t}}$

$\frac{e^{2u} + 4e^{u}}{e^{u}}$

$\sqrt{e^2 + 4}$

b) $K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

$= \frac{\left\| \begin{pmatrix} 2 \\ e^t - e^{-t} \end{pmatrix} \times \begin{pmatrix} 0 \\ e^t + e^{-t} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} 2 \\ e^t - e^{-t} \end{pmatrix} \right\|^3} = \quad \begin{pmatrix} 2 \\ e^t - e^{-t} \end{pmatrix} \times \begin{pmatrix} 0 \\ e^t + e^{-t} \end{pmatrix} =$

$2e^t + 2e^{-t}\,\vec{i} - 0\,\vec{j} = 2e^t + 2e^{-t}\,\vec{i} \quad = \begin{pmatrix} 2e^t + 2e^{-t} \\ 0 \end{pmatrix}$

same mistake as above

$\left\| \begin{pmatrix} 2e^t + 2e^{-t} \\ 0 \end{pmatrix} \right\| = \sqrt{(2e^t + 2e^{-t})^2 + 0^2} = \sqrt{4e^{2t} + \cancel{8e^{t-t}} + 4e^{-2t}} = \sqrt{4e^{2t} + 4e^{-2t}}$

$\left\| \begin{pmatrix} 2 \\ e^t - e^{-t} \end{pmatrix} \right\|^3 = \left(\sqrt{2^2 + (e^t - e^{-t})^2}\right)^3 = \left(\sqrt{4 + e^{2t} - \cancel{2e^{t-t}} + e^{-2t}}\right)^3 = \left(\sqrt{4 + e^{2t} + e^{-2t}}\right)^3$

$= 4 + e^{2t} + e^{-2t}\sqrt{4 + e^{2t} + e^{-2t}}$

$\boxed{K(t) = \frac{\sqrt{4e^{2t} + 4e^{-2t}}}{4 + e^{2t} + e^{-2t}\,\sqrt{4 + e^{2t} + e^{-2t}}}}$