1. Let curve C be the intersection of the surfaces

$$x^2+y^2=z^2, \quad y=z^2.$$

$a)$ (6 pts) Parametrize C by using one of the variables x, y, or z as the parameter.

$b)$ (4 pts) Show that C lies on the surface of a sphere of radius 1.

a) $y=z^2$, $x^2+y^2=z^2$;

$x^2 = z^2-y^2 = z^2-(z^2)^2 = z^2-z^4$

$\therefore$ $z=z$, $y=z^2$ & $x=\pm\sqrt{z^2-z^4}$, if $z=t$: 4

So $C_1(t)=\langle +\sqrt{t^2-t^4},\ t^2,\ t\rangle$ (above xy plane. ~~+ve area~~)

$C_2(t)=\langle -\sqrt{t^2-t^4},\ t^2,\ t\rangle$ (below xy plane. ~~-ve area~~)

b). Now, eq^n of a sphere is given by

$e(R) = \langle R\cos\theta,\ R\sin\theta,\ R\rangle$ It's a different sphere.

Horizontal traces of a sphere are elipses.

i.e. consider horizontal trace of c_1 & c_2 at $f(x,y)=0$

Traces: $\langle t\sqrt{1-t^2},\ t^2,\ 0\rangle$.

(0,0)

Now ~~if this is a circle at any pt. it's~~ part of a sphere.

$\cancel{x^2+y^2=R^2}$: $t^2(1-t^2)+t^2=R^2$

$\cancel{t^2-t^4+t^2=R^2}$

$2t^2-t^4=R^2$

~~If~~ Comparing

sphere: $e(R) = \langle R\cos\theta,\ R\sin\theta,\ R\rangle = R\langle\cos\theta, \sin\theta, 1\rangle$

& $C = \langle t\sqrt{1-t^2},\ t^2,\ t\rangle = t\langle\sqrt{1-t^2},\ t,\ 1\rangle$

If $R=t$: $\cos\theta=\sqrt{1-t^2}$ & $\sin\theta=t$, $(\sqrt{1-t^2})^2+(t)^2=\underline{1}$.

Since this follows $\sin\theta$ & $\cos\theta$.

the comparison matches & lies on a sphere of radius 1.

① diff & int for s
② take inverse.
③ substituted.

2. (10 pts) Find an arc length parametrization of the line

$$y = mx + k,$$

where m and k are arbitrary real numbers.

$y = mx + k$

$\therefore$ Line : $\bar{r}(t) = \langle t, mt + k \rangle$.

$\bar{r}'(t) = \langle 1, m \rangle$.

$\|\bar{r}'(t)\| = \sqrt{1+m^2}$

Let $\overset{\text{arc length}}{s} = \int_0^t \|\bar{r}'(t)\| dt = \int_0^t \sqrt{1+m^2}\, dt$

$s = \sqrt{1+m^2}\, t$.

$\Rightarrow t = \frac{s}{\sqrt{1+m^2}}$

Substituting to get arc length parameterization in $\bar{r}(t)$ as $\bar{r}_1(s)$:

$\bar{r}_1(s) = \langle \frac{s}{\sqrt{1+m^2}}, \frac{ms}{\sqrt{1+m^2}} + k \rangle$.

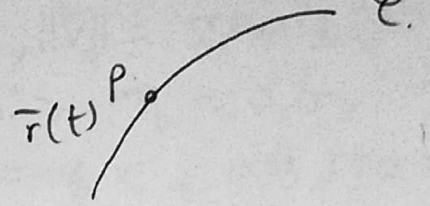
10

3. *a*) (2 pts) A particle P is moving along a curve C according to a parametrization $\mathbf{r}(t)$. At time t_0, the particle P is accelerating in the direction opposite from its current velocity. Find the curvature of C at the point $\mathbf{r}(t_0)$.

b) (2 pts) A particle Q is moving along a curve D according to a parametrization $\mathbf{r}(t)$. At time t_0, the tangential component of the acceleration of Q is equal to 0, but the acceleration itself is nonzero. What can you say about the direction of the acceleration of Q relative to the velocity of Q at time t_0?

c) (6 pts) Decompose the acceleration vector $\mathbf{a}$ of $\mathbf{r}(t) = \langle t^2, 2t, \ln t\rangle$ into tangential and normal components at $t = \frac{1}{2}$.

a). ~~At~~ t_0, $\bar{a} = \bar{r}''(t)$ is $\uparrow\downarrow$ to $\bar{v} = \bar{r}'(t)$



2

$\therefore$ ~~$\bar{a}\times\bar{v}$~~ $\bar{v}\times\bar{a} = 0$.

$\Rightarrow K = \dfrac{\|\bar{r}'(t_0)\times\bar{r}''(t_0)\|}{\|\bar{r}'(t_0)\|^3} = 0$ direct

$-4+\frac{4}{3} = \frac{-12+4}{3} = \frac{-8}{3}$

$\frac{2}{3} + 2$
$= \frac{8}{3}$

b). $a_T\bar{T} = 0$ at t_0
$\bar{a}\neq 0$ @ t_0 $\bar{a} = a_T\bar{T} + a_N\bar{N}$
$\bar{a} = a_N\bar{N}$

10

64
64
16
144

2

$\therefore$ acc^n only contains normal component which is ~~in~~ $\perp$ to movement (velocity).

c). $\bar{r}(t) = \langle t^2, 2t, \ln t\rangle$ | $\bar{a} = a_N\bar{N} + a_T\bar{T}$

$\bar{r}'(t) = \langle 2t, 2, \frac{1}{t}\rangle$

$\bar{r}'(\frac{1}{2}) = \langle 1, 2, 2\rangle$; $\|\bar{r}'(\frac{1}{2})\| = \sqrt{1^2+2^2+2^2} = 3$. (mag is true)

6

$\bar{T}(t) = \dfrac{\bar{r}'(t)}{\|\bar{r}'(t)\|}$; $\bar{T}(\frac{1}{2}) = \frac{1}{3}\cdot\langle 1, 2, 2\rangle$

$\bar{a}(t) = \bar{r}''(t) = \langle 2, 0, \frac{-1}{t^2}\rangle$, $\bar{a}(\frac{1}{2}) = \langle 2, 0, -4\rangle$

$a_T = \bar{a}(\frac{1}{2})\cdot\bar{T}(\frac{1}{2}) = \langle 2,0,-4\rangle\cdot\langle\frac{1}{3},\frac{2}{3},\frac{2}{3}\rangle = \frac{2}{3} - \frac{8}{3} = \frac{-6}{3} = \underline{\underline{-2}}$

$\therefore a_T\bar{T} = -2\langle\frac{1}{3},\frac{2}{3},\frac{2}{3}\rangle = \langle\frac{-2}{3},\frac{-4}{3},\frac{-4}{3}\rangle$

$a_N\bar{N} = \bar{a} - a_T\bar{T} = \langle 2,0,-4\rangle - \langle\frac{-2}{3},\frac{-4}{3},\frac{-4}{3}\rangle = \langle\frac{8}{3},\frac{4}{3},\frac{-8}{3}\rangle$

$a_N = \|a_N\bar{N}\| = \sqrt{\frac{64}{9}+\frac{64}{9}+\frac{16}{9}} = \frac{1}{3}\sqrt{144} = \frac{1}{3}\cdot 12 = 4$ //

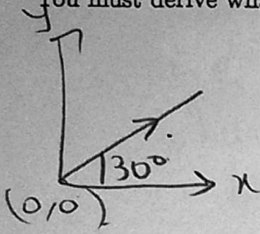
$\Rightarrow \bar{N} = \langle\frac{2}{3},\frac{1}{3},-2/3\rangle \Rightarrow \boxed{\bar{a} = 4\bar{N} - 2\bar{T}}$ where $\bar{N} = \frac{1}{3}\langle 2,1,-2\rangle$, $\bar{T} = \frac{1}{3}\langle 1,2,2\rangle$

4. A bullet is fired at an angle of 30 degrees above the horizontal, with initial speed of 500 m/s. You may approximate the force of gravity as 10 m/s^2 and ignore air resistance on the bullet.

a) (5 pts) How long after being fired will the bullet reach its maximum height?

b) (5 pts) How far will the bullet have traveled relative to the ground when it reaches its maximum height?

You must derive whatever formulas you employ by using the techniques we have learned in this course.



$\bar{a} = \langle 0, -10 \rangle \ m/s^2$; $\|\bar{v}_0\| = 500 \ m/s$

$\bar{v}_0 = \|\bar{v}_0\| \cdot \langle \cos 30°, \sin 30° \rangle$; $\bar{r}_0 = \langle 0, 0 \rangle$

$= 500 \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \langle 250\sqrt{3}, 250 \rangle \ m/s$

$\bar{v}(t) = \int_0^t \bar{a}(t)\,dt = \langle 0, -10t \rangle + C_1$

$\bar{v}(0) = \bar{v}_0 \Rightarrow \langle 0, 0 \rangle + C_1 = \langle 250\sqrt{3}, 250 \rangle$.

$C_1 = \langle 250\sqrt{3}, 250 \rangle$ & $\bar{v}(t) = \langle 250\sqrt{3}, 250 - 10t \rangle$ m/s.

$\bar{r}(t) = \int_0^t \bar{v}(t)\,dt = \langle 250\sqrt{3}t, 250t - 5t^2 \rangle + C_2$ m.

$\bar{r}(0) = r_0 \Rightarrow \langle 0, 0 \rangle + C_2 = \langle 0, 0 \rangle$

$\therefore C_2 = 0$

$\Rightarrow \bar{r}(t) = \langle 250\sqrt{3}t, 250t - 5t^2 \rangle$

Taking y component: Now height is max when $y(t) = 250t - 5t^2$ is maximum.

$y'(t) = 250 - 10t$, $y''(t) = -10 < 0 \Rightarrow$ found is maxima

Set to zero: $t = 250/10 = 25$ s.

a). It will reach max height after 25 s

b). Max distance will be given by x component of $\bar{r}(25)$

$\bar{r}(25) = \langle 250\sqrt{3} \cdot 25, 250 \cdot 25 - 5(25)^2 \rangle$.

$x_{\text{travelled}} = 6250\sqrt{3}$ m

625

$\frac{x^2+y^2}{1+xy}$.

5. Evaluate the limit or show that it does not exist.

a) (5 pts) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy^2}$. 5

b) (5 pts) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{1+xy^2}$. 5 $\frac{x^3+x^6}{1+x^5} = 0$. x^3+y^3

a) Consider ~~any~~ eq^n of a general line through $(0,0)$ i.e $y=mx$. For the limit to exist, the limit must have value independent of m $\because$ ~~it~~ can vary from line to line.

$$\lim_{(x,mx)\to(0,0)} \frac{x^3+y^3}{xy^2} = \lim_{(x,mx)\to(0,0)} \frac{x^3+m^3x^3}{m^2x^3}$$

$$= \lim_{(x,mx)\to(0,0)} \frac{1+m^3}{m^2}$$

$$= \frac{1}{m^2} + m.$$

Now, this value clearly depends on m. So lines of different slopes have different values for this limit, hence it does not exist

b) $\lim_{(x,0)\to(0,0)} \frac{x^3+y^3}{1+xy^2} = \lim_{(x,0)\to(0,0)} \frac{x^3}{1} = \lim_{(x,0)\to(0,0)} x^3 = 0$.

$\lim_{(0,y)\to(0,0)} \frac{x^3+y^3}{1+xy^2} = \lim_{y\to 0} \frac{y^3}{1+0} = \lim_{(0,y)\to(0,0)} y^3$.

Let $x = r\cos\theta$, $y = r\sin\theta$.

$$\lim_{(r\cos\theta, r\sin\theta)\to(0,0)} \frac{x^3+y^3}{1+xy^2} = \frac{r^3(\sin^3\theta+\cos^3\theta)}{1+r^3(\sin^2\theta\cos\theta)}$$

If $y=mx$.

$$\lim_{(x,mx)\to(0,0)} \frac{x^3+y^3}{1+xy^2} = \lim_{(x,mx)\to(0,0)} \frac{x^3+m^3x^3}{1+m^2x^3} = \lim_{(x,mx)\to(0,0)} 0$$

why? So this is cts $\because$ $f(x,y)$ @ $f(0,0) = \lim_{(x,y)\to(0,0)} f(x)$

Thus limit is zero