Math 32A
Summer Session C 2018
Friday, September 14, 2018
Time Limit: 10:00 until 11:50

Name: ______________________
UID: ______________________

Final Exam

- **DO NOT open the exam booklet until you are told to begin. You should write your name and section number at the top and read the instructions.**
- Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly.

Problem	Points	Score
1	30	
2	30	
3	40	
4	30	
5	30	
6	30	
7	30	
Total:	220	

1. (30 points) Find an arc length parametrization of the curve

$$\mathbf{r}(t) = \left\langle 2t^{3/2}, t, \frac{4}{\sqrt{3}}t^{3/2} \right\rangle$$

with the parameter s measuring arc length along the curve from $(0,0,0)$. *Hint:* you may need to perform a u-substitution at some point.

Solution:

We have

$$\begin{aligned} \mathbf{r}'(t) &= \langle 3t^{1/2}, 1, 2\sqrt{3}t^{1/2} \rangle \\ \|\mathbf{r}'(t)\| &= \sqrt{(3t^{1/2})^2 + 1^2 + (2\sqrt{3}t^{1/2})^2} \\ &= \sqrt{9t + 1 + 12t} \\ &= \sqrt{21t + 1} \end{aligned}$$

Therefore the arc length function is given by

$$\begin{aligned} s = g(t) &= \int_0^t \|\mathbf{r}'(u)\| \, \mathrm{d}u \\ &= \int_0^t \sqrt{21u + 1} \, \mathrm{d}u \\ &= \frac{1}{21} \cdot \frac{2}{3} (21u + 1)^{3/2} \Big|_{u=0}^{u=t} \\ &= \frac{2}{63} \left[(21t + 1)^{3/2} - 1 \right] \end{aligned}$$

Solving for t in terms of s, we obtain:

$$t = \frac{\left(\frac{63s}{2} + 1\right)^{2/3} - 1}{21}$$

and so the arc length parametrization is:

$$\boxed{\mathbf{r}_1(s) = \left\langle 2\left[\frac{(\frac{63s}{2} + 1)^{2/3} - 1}{21}\right]^{3/2}, \frac{(\frac{63s}{2} + 1)^{2/3} - 1}{21}, \frac{4}{\sqrt{3}}\left[\frac{(\frac{63s}{2} + 1)^{2/3} - 1}{21}\right]^{3/2} \right\rangle}$$

2. (30 points) Find the Frenet frame ($\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$) to the curve parametrized by

$$\mathbf{r}(t) = \langle \sin(2t),\, e^t,\, \cos(2t) \rangle$$

at the point $(0, 1, 1)$.

Solution: First observe that the point $(0,1,1)$ corresponds to $t = 0$. We have

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 2\cos(2t), e^t, -2\sin(2t) \rangle \\
\|\mathbf{r}'(t)\| &= \sqrt{(2\cos(2t))^2 + (e^t)^2 + (-2\sin(2t))^2} \\
&= \sqrt{4\cos^2(2t) + e^{2t} + 4\sin^2(2t)} \\
&= \sqrt{4 + e^{2t}} \\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\
&= \frac{\langle 2\cos(2t), e^t, -2\sin(2t) \rangle}{\sqrt{4+e^{2t}}} \\
\mathbf{T}(0) &= \boxed{\frac{\langle 2,1,0 \rangle}{\sqrt{5}}}
\end{aligned}$$

Taking the derivative of $\mathbf{T}(t)$, we obtain:

$$\begin{aligned}
\mathbf{T}'(t) &= \frac{\langle -4\sin(2t), e^t, -4\cos(2t) \rangle \sqrt{4+e^{2t}} - \frac{1}{2}(4+e^{2t})^{-1/2} \cdot 2e^{2t} \langle 2\cos(2t), e^t, -2\sin(2t) \rangle}{4+e^{2t}} \\
\mathbf{T}'(0) &= \frac{\sqrt{5}\langle 0,1,-4 \rangle - 5^{-1/2}\langle 2,1,0 \rangle}{5}
\end{aligned}$$

Now $\mathbf{N}(0)$ will be the unit normalization of $\mathbf{T}'(0)$. Since the unit normalization of a vector is equal to that of any of its positive scalar multiples, $\mathbf{N}(0)$ is also the unit normalization of

$$\begin{aligned}
5\sqrt{5}\mathbf{T}'(0) &= 5\langle 0,1,-4 \rangle - \langle 2,1,0 \rangle \\
&= \langle 0,5,-20 \rangle - \langle 2,1,0 \rangle \\
&= \langle -2,4,-20 \rangle
\end{aligned}$$

Since we see that all components of this vector are divisible by 2, in fact we will compute the unit normalization of

$$\frac{5\sqrt{5}}{2}\mathbf{T}'(0) = \langle -1, 2, -10 \rangle$$

This gives

$$\begin{aligned}
\mathbf{N}(0) &= \frac{\langle -1, 2, -10\rangle}{\|\langle -1, 2, -10\rangle\|} \\
&= \frac{\langle -1, 2, -10\rangle}{\sqrt{1+4+100}} \\
&= \boxed{\frac{\langle -1, 2, -10\rangle}{\sqrt{105}}}
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathbf{B}(0) &= \mathbf{T}(0) \times \mathbf{N}(0) \\
&= \frac{1}{\sqrt{5}\sqrt{105}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -1 & 2 & -10 \end{vmatrix} \\
&= \frac{1}{\sqrt{5}\sqrt{5}\sqrt{21}} \langle -10, 20, 5\rangle \\
&= \frac{1}{5\sqrt{21}} \langle -10, 20, 5\rangle \\
&= \boxed{\frac{\langle -2, 4, 1\rangle}{\sqrt{21}}}
\end{aligned}$$

3. Compute the following limits or show that they do not exist:

(a) (10 points)

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^4}{x^8+y^8}$$

Solution: Taking the limit along the curve $y = 0$ (the x-axis), we obtain:

$$\lim_{x\to 0} \frac{(x^2)^4}{x^8} = \lim_{x\to 0} \frac{x^8}{x^8} = 1$$

Taking the limit along the curve $y = x$ we obtain:

$$\lim_{x\to 0} \frac{(x^2+x^2)^4}{x^8+x^8} = \lim_{x\to 0} \frac{(2x^2)^4}{2x^8} = \lim_{x\to 0} \frac{16x^8}{2x^8} = 8$$

Since the two limits disagree, we conclude that the original limit does not exist.

(b) (10 points)

$$\lim_{(x,y)\to(0,0)} \frac{7x^{10}y}{4x^{15}+2y^3}$$

Solution: Taking the limit along the curve $y = 0$ we obtain:

$$\lim_{x\to 0} \frac{0}{4x^{15}+0} = 0$$

Taking the limit along the curve $y = x^5$, we obtain:

$$\lim_{x\to 0} \frac{7x^{10}\cdot x^5}{4x^{15}+2(x^5)^3} = \lim_{x\to 0} \frac{7x^{15}}{6x^{15}} = \frac{7}{6}$$

Since the two limits disagree, we conclude that the original limit does not exist.

(c) (10 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^4+y^4+1)^{1/3}-1}$$

Hint: Recall the identity $A^3 - B^3 = (A-B)(A^2+AB+B^2)$

Solution: Following the hint, we have:

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^4+y^4+1)^{1/3}-1} \\
&= \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^4+y^4+1)^{1/3}-1} \cdot \frac{(x^4+y^4+1)^{2/3}+(x^4+y^4+1)^{1/3}+1}{(x^4+y^4+1)^{2/3}+(x^4+y^4+1)^{1/3}+1} \\
&= \lim_{(x,y)\to(0,0)} \frac{(x^4+y^4)[(x^4+y^4+1)^{2/3}+(x^4+y^4+1)^{1/3}+1]}{x^4+y^4+1-1^3} \\
&= \lim_{(x,y)\to(0,0)} (x^4+y^4+1)^{2/3}+(x^4+y^4+1)^{1/3}+1 \\
&= 3
\end{aligned}$$

(d) (10 points)

$$\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}\sin\left(\ln\left(\frac{1}{1+x^6+17y^8}\right)\right)$$

Solution: We have:

$$0 \le \left|\frac{x\sin(x^2+y^2)}{x^2+y^2}\sin\left(\ln\left(\frac{1}{1+x^6+17y^8}\right)\right)\right| \le \left|\frac{x\sin(x^2+y^2)}{x^2+y^2}\right|$$

and so if we can verify

$$\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2} = 0$$

then the original limit will also be 0 by the Squeeze Theorem. For the computation of the latter limit, observe:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2} &= \left(\lim_{(x,y)\to(0,0)} x\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}\right) \\
&= \left(\lim_{x\to 0} x\right)\left(\lim_{r\to 0} \frac{\sin(r^2)}{r^2}\right) \\
&= 0\cdot 1 \\
&= 0
\end{aligned}$$

where we have converted to polar coordinates in the second factor.

4. (a) (10 points) Find the equation of the tangent plane to the surface

$$-x^2 - 9y^2 + 25z^2 = 4$$

at the point $P = \left(2\sqrt{3}, 1, 1\right)$

Solution: The surface is of the form $F(x, y, z) = 4$ for the function

$$F(x, y, z) = -x^2 - 9y^2 + 25z^2$$

We have

$$\begin{aligned} \nabla F &= \langle -2x, -18y, 50z \rangle \\ \nabla F\big|_P &= \langle -4\sqrt{3}, -18, 50 \rangle \end{aligned}$$

The equation for the tangent plane is

$$\nabla F\big|_P \cdot (\langle x, y, z \rangle - P) = 0$$

or

$$\begin{aligned} \langle -4\sqrt{3}, -18, 50 \rangle \cdot \langle x - 2\sqrt{3}, y - 1, z - 1 \rangle &= 0 \\ -4\sqrt{3}(x - 2\sqrt{3}) - 18(y - 1) + 50(z - 1) &= 0 \\ -2\sqrt{3}(x - 2\sqrt{3}) - 9(y - 1) + 25(z - 1) &= 0 \\ -2\sqrt{3}x + 12 - 9y + 9 + 25z - 25 &= 0 \\ \boxed{-2\sqrt{3}x - 9y + 25z = 4} \end{aligned}$$

(b) (20 points) Find all points on the surface

$$-x^2 - 9y^2 + 25z^2 = 4$$

where the tangent plane is orthogonal to the vector $\mathbf{v} = \langle 2, 6, 20 \rangle$

Solution: Let $F(x, y, z)$ be as in part (a). Again, we have

$$\nabla F = \langle -2x, -18y, 50z \rangle$$

The tangent plane at a point (x, y, z) on the surface will be orthogonal to $\mathbf{v}$ when $\nabla F = \lambda \mathbf{v}$ for some scalar λ. In other words, when:

$$\begin{aligned} -2x &= 2\lambda \\ -18y &= 6\lambda \\ 50z &= 20\lambda \end{aligned}$$

Solving these equations, we get

$$\begin{aligned} x &= -\lambda \\ y &= -\frac{1}{3}\lambda \\ z &= \frac{2}{5}\lambda \end{aligned}$$

and substituting these equations into the equation of the surface we get

$$\begin{aligned} -(-\lambda)^2 - 9\left(-\frac{1}{3}\lambda\right)^2 + 25\left(\frac{2}{5}\lambda\right)^2 &= 4 \\ -\lambda^2 - \lambda^2 + 4\lambda^2 &= 4 \\ 2\lambda^2 &= 4 \\ \lambda^2 &= 2 \\ \lambda &= \pm\sqrt{2} \end{aligned}$$

and so the two points are

$$P_1 = \boxed{\left(-\sqrt{2}, -\frac{\sqrt{2}}{3}, \frac{2\sqrt{2}}{5}\right)}$$

$$P_2 = \boxed{\left(\sqrt{2}, \frac{\sqrt{2}}{3}, -\frac{2\sqrt{2}}{5}\right)}$$

5. Consider the function $f(x,y) = 2x^2 + 3y^3$. The aim of this problem is to show that $f(x,y)$ is differentiable at $(2,1)$ by using the definition of differentiability directly.

(a) (10 points) Compute the linearization $L(x,y)$ at the point $(2,1)$.

Solution: We have

$$\begin{aligned} f_x(x,y) &= 4x \\ f_y(x,y) &= 9y^2 \end{aligned}$$

So

$$\begin{aligned} L(x,y) &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 11 + 8(x-2) + 9(y-1) \\ &= \boxed{8x + 9y - 14} \end{aligned}$$

(b) (10 points) Recall that $e(x,y) = f(x,y) - L(x,y)$ gives the error between a function and its linear approximation. Show that $e(x,y) = 2(x-2)^2 + 3(y-1)^2(y+2)$.

Solution: We have:

$$\begin{aligned} e(x,y) &= f(x,y) - L(x,y) \\ &= 2x^2 + 3y^3 - (8x + 9y - 14) \\ &= 2x^2 - 8x + 3y^3 - 9y + 14 \end{aligned}$$

and

$$\begin{aligned} 2(x-2)^2 + 3(y-1)^2(y+2) &= 2(x^2 - 4x + 4) + 3(y^2 - 2y + 1)(y+2) \\ &= 2x^2 - 8x + 8 + 3(y^3 - 2y^2 + y + 2y^2 - 4y + 2) \\ &= 2x^2 - 8x + 8 + 3(y^3 - 3y + 2) \\ &= 2x^2 - 8x + 8 + 3y^3 - 9y + 6 \\ &= 2x^2 - 8x + 3y^3 - 9y + 14 \end{aligned}$$

so the two quantities are equal.

(c) (10 points) Recall that $f(x, y)$ is differentiable at (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{e(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

Use this definition in conjunction with the two previous parts to show that $f(x, y)$ is differentiable at $(2, 1)$.

Solution: We have

$$\lim_{(x,y)\to(2,1)} \frac{e(x,y)}{\sqrt{(x-2)^2+(y-1)^2}} = \lim_{(x,y)\to(2,1)} \frac{2(x-2)^2}{\sqrt{(x-2)^2+(y-1)^2}} + \lim_{(x,y)\to(2,1)} \frac{3(y-1)^2(y+2)}{\sqrt{(x-2)^2+(y-1)^2}}$$

so it suffices to show that each of the summands on the right hand side approaches zero as $(x, y) \to (2, 1)$. For the first, note

$$0 \le \frac{2(x-2)^2}{\sqrt{(x-2)^2+(y-1)^2}} \le \frac{2(x-2)^2}{\sqrt{x-2}^2} = \frac{2(x-2)^2}{|x-2|} = 2\,|x-2|$$

and since $\lim_{(x,y)\to(2,1)} 2\,|x-2| = \lim_{x\to 2} 2\,|x-2| = 0$ we may conclude

$$\lim_{(x,y)\to(2,1)} \frac{2(x-2)^2}{\sqrt{(x-2)^2+(y-1)^2}} = 0$$

by the Squeeze Theorem. For the second, note that

$$0 \le \left| \frac{3(y-1)^2(y+2)}{\sqrt{(x-2)^2+(y-1)^2}} \right| \le \left| \frac{3(y-1)^2(y+2)}{y-1} \right| = 3\,|(y-1)(y+2)|$$

and since $\lim_{(x,y)\to(2,1)} 3\,|(y-1)(y+2)| = \lim_{y\to 1} 3\,|(y-1)(y+2)| = 0$ we may conclude (again using the Squeeze Theorem) that

$$\lim_{(x,y)\to(2,1)} \left| \frac{3(y-1)^2(y+2)}{\sqrt{(x-2)^2+(y-1)^2}} \right| = 0$$

and hence

$$\lim_{(x,y)\to(2,1)} \frac{3(y-1)^2(y+2)}{\sqrt{(x-2)^2+(y-1)^2}} = 0$$

6. (30 points) Find all critical points of the function

$$f(x, y) = x^3 + 3xy + xy^2$$

and use the Second Derivative Test to classify them as local minima, local maxima, or saddle points (or state that the test fails).

Solution: We have

$$f_x(x, y) = 3x^2 + 3y + y^2$$
$$f_y(x, y) = 3x + 2xy = x(3 + 2y)$$

Setting the latter equal to 0 we have either $x = 0$ or $y = -3/2$. If $x = 0$, then setting $f_x(x, y)$ equal to zero gives

$$3y + y^2 = 0$$
$$y(3 + y) = 0$$

so $y = 0$ or $y = -3$.

If $y = -3/2$, setting $f_x(x, y)$ equal to zero gives

$$3x^2 + 3\left(-\frac{3}{2}\right) + \left(-\frac{3}{2}\right)^2 = 0$$
$$3x^2 - \frac{9}{4} = 0$$
$$x^2 = \frac{3}{4}$$
$$x = \pm\frac{\sqrt{3}}{2}$$

Therefore the critical points are $(0, 0)$, $(0, -3)$, $(\sqrt{3}/2, -3/2)$, and $(-\sqrt{3}/2, -3/2)$. The discriminant is given by

$$D(x, y) = \begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{vmatrix} = \begin{vmatrix} 6x & 3 + 2y \\ 3 + 2y & 2x \end{vmatrix}$$

We have:

- $$D(0, 0) = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0$$

  so $(0, 0)$ is a saddle point.

- $$D(0, -3) = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0$$

  so $(0, -3)$ is also a saddle point.

- $$D\left(\frac{\sqrt{3}}{2}, -\frac{3}{2}\right) = \begin{vmatrix} 3\sqrt{3} & 0 \\ 0 & \sqrt{3} \end{vmatrix} = 9 > 0$$

  and $f_{xx}\left(\frac{\sqrt{3}}{2}, -\frac{3}{2}\right) = 3\sqrt{3} > 0$, so $\left(\frac{\sqrt{3}}{2}, -\frac{3}{2}\right)$ is a local minimum.

- $$D\left(-\frac{\sqrt{3}}{2}, -\frac{3}{2}\right) = \begin{vmatrix} -3\sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{vmatrix} = 9 > 0$$

  and $f_{xx}\left(-\frac{\sqrt{3}}{2}, -\frac{3}{2}\right) = -3\sqrt{3} < 0$, so $\left(\frac{-\sqrt{3}}{2}, -\frac{3}{2}\right)$ is a local maximum.

7. (a) (10 points) Suppose that p, q are positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Use Lagrange multipliers to show that the minimum value of

$$f(x,y)=\frac{x^p}{p}+\frac{y^q}{q}$$

over the curve $xy=C$, $x,y>0$ (for C some positive real constant) is C. Conclude that *Young's Inequality*

$$xy\leq\frac{x^p}{p}+\frac{y^q}{q}$$

holds for any $x,y>0$ and p, q as above.

Solution: First observe that $\frac{1}{p}+\frac{1}{q}=1$ is equivalent to the condition $p+q=pq$ (as may be observed by mulitplying both sides by pq).

We are trying to minimize $f(x,y)$ subject to the constraint $g(x,y)=C$, where $g(x,y)=xy$. By the method of Lagrange multipliers, the extreme value occurs when

$$\nabla f=\lambda\nabla g$$

for some scalar λ. In other words,

$$\begin{aligned}x^{p-1}&=\lambda y\\ y^{q-1}&=\lambda x\end{aligned}$$

This gives

$$\frac{x^{p-1}}{y}=\lambda=\frac{y^{q-1}}{x}$$

and cross-multiplying gives $x^p=y^q$, or $y=x^{p/q}$. Substituting this into the constraint $xy=C$ gives

$$\begin{aligned}x\cdot x^{\frac{p}{q}}&=C\\ x^{\frac{p+q}{q}}&=C\\ x&=C^{\frac{q}{p+q}}=C^{\frac{q}{pq}}=C^{\frac{1}{p}}\end{aligned}$$

where we have invoked the assumption $p+q=pq$. Therefore

$$y=x^{\frac{p}{q}}=\left(C^{\frac{1}{p}}\right)^{\frac{p}{q}}=C^{\frac{1}{q}}$$

For these values of x and y we have:

$$f(x,y)=\frac{\left(C^{1/p}\right)^p}{p}+\frac{\left(C^{1/q}\right)^q}{q}=\frac{C}{p}+\frac{C}{q}=\left(\frac{1}{p}+\frac{1}{q}\right)C=C$$

Thus the extreme value occurs when

$$\frac{x^p}{p} + \frac{y^q}{q} = C = xy$$

To show that this is the minimum rather than the maximum observe that there are values on the constraint curve $xy = C$ where y is arbitrarily large, and therefore there are values where $f(x, y)$ is arbitrarily large as well.

(b) (20 points) Use a similar method to prove the Arithmetic Mean-Harmonic Mean Inequality:

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \leq \frac{x_1+x_2+\cdots+x_n}{n}$$

for any positive real numbers $x_1, x_2, \ldots, x_n$. *Bonus:* Can you prove this any other way?

Solution: Let us minimize

$$f(x_1,\ldots,x_n) = \frac{x_1+x_1+\cdots+x_n}{n}$$

subject to $g(x_1,\ldots,x_n) = C$, where

$$g(x_1,\ldots,x_n) = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots\frac{1}{x_n}}$$

By the method of Lagrange multipliers, the extreme value occurs when $\nabla f = \lambda \nabla g$ for some scalar λ, or:

$$\frac{1}{n} = \lambda \cdot \frac{-n}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^2} \cdot \left(-\frac{1}{x_1^2}\right)$$

$$\frac{1}{n} = \lambda \cdot \frac{-n}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^2} \cdot \left(-\frac{1}{x_2^2}\right)$$

$$\vdots$$

$$\frac{1}{n} = \lambda \cdot \frac{-n}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^2} \cdot \left(-\frac{1}{x_n^2}\right)$$

Therefore

$$x_i^2 = \frac{\lambda n^2}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^2}$$

for each $1 \leq i \leq n$, and so in particular $x_1^2 = x_2^2 = \cdots = x_n^2$. Since each x_i was assumed positive, we have $x_1 = x_2 = \cdots = x_n$. Using this in conjunction with the constraint $g(x_1,\ldots,x_n) = C$, we have

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_1}+\cdots+\frac{1}{x_1}} = C$$

$$\frac{n}{n\left(\frac{1}{x_1}\right)} = C$$

$$x_1 = C$$

and so each x_i equals C. For this value of the x_i's we have

$$f(x_1, \dots, x_n) = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{C + C + \dots + C}{n} = \frac{nC}{n} = C$$

Thus the extreme value occurs when

$$\frac{x_1 + x_2 + \dots + x_n}{n} = C = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}}$$

To see that this is a minimum rather than a maximum, observe that there are values on the constraint curve where x_1 is arbitrarily large, and so $f(x_1, \dots, x_n)$ may be arbitrarily large as well.

Solution to Bonus: The inequality we wish to show is equivalent to:

$$\left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) \geq n^2$$

We have:

$$\begin{aligned}
\left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{x_i}{x_j} \\
&= \sum_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right) + \sum_{i=1}^{n} \frac{x_i}{x_i} \\
&\geq \sum_{1 \leq i < j \leq n} 2 + \sum_{i=1}^{n} 1 \\
&= 2 \binom{n}{2} + n \\
&= 2 \left(\frac{n(n-1)}{2} \right) + n \\
&= n^2
\end{aligned}$$

as desired. Here we've used the inequality

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

which comes from expanding the inequality

$$(a - b)^2 \geq 0$$

and dividing by ab.