20W-MATH32A-3 Final

TOTAL POINTS

183.25 / 200

QUESTION 1

30 pts

1.1 a **4 / 5**

✓ **- 1 pts** **Common mistake: $$\frac{\partial}{\partial y} e^{x^2 y^2} = 2y e^{x^2y^2}$$**

1.2 b **7.5 / 8**

✓ **- 0.5 pts** **Error copying**

1.3 c **8 / 8**

✓ **- 0 pts** **Correct**

1.4 d **8 / 9**

✓ **- 1 pts** **Sign error: $$6+ 4 = 10$$**

QUESTION 2

15 pts

2.1 a **8 / 8**

✓ **- 0 pts** **Correct**

2.2 b **7 / 7**

✓ **- 0 pts** **Correct**

QUESTION 3

20 pts

3.1 a **7 / 7**

✓ **- 0 pts** **Correct**

3.2 b **13 / 13**

✓ **- 0 pts** **Correct**

QUESTION 4

4 **9.75 / 10**

- 0.25 **Point adjustment**

What is t??

QUESTION 5

5 **20 / 20**

✓ **+ 20 pts** **Correct**

+ **2 pts** Calculate x partial derivative
+ **2 pts** Calculate y partial derivative
+ **2 pts** Found critical point (0,0)
+ **2 pts** Found critical point (-2,2)
+ **1 pts** Calculate xx partial derivative
+ **1 pts** Calculate xy partial derivative
+ **1 pts** Calculate yy partial derivative
+ **2 pts** Calculate Hessian determinant for (0,0)
+ **2 pts** Classify (0,0) as a saddle point
+ **2 pts** Calculate Hessian determinant for (-2,2)
+ **1 pts** Check sign of f_xx(-2, 2) or f_yy(-2,2)
+ **2 pts** Classify (-2,2) as a local minimum

QUESTION 6

6 **20 / 20**

✓ **+ 20 pts** **Correct**

+ **1 pts** Check for max/min in the interior: Calculated x derivative
+ **1 pts** Check for max/min in the interior: Calculated y derivative
+ **1 pts** Check for max/min in the interior: Found the point (10,6)
+ **1 pts** Check for max/min in the interior: Deduced that (10,6) is not in the domain
+ **2 pts** Check for max/min on left boundary: Found max of 15 at (0, 5)
+ **2 pts** Check for max/min on left boundary: Found min of -1 at (0, 1)
+ **2 pts** Check for max/min on top boundary: Found max of 20 at (5, 5)
+ **2 pts** Check for max/min on top boundary: Found

min of 15 at (0, 5)

+ **2 pts** Check for max/min on diagonal boundary: Found max of 20 at (5, 5)

+ **2 pts** Check for max/min on diagonal boundary: Found min of 0 at (0, 0)

+ **2 pts** Found global max of 20 at (5, 5)

+ **2 pts** Found global min of -1 at (0, 1)

+ **0 pts** No Credit and/or blank

QUESTION 7

7 15 / 15

✓ **- 0 pts** Correct

QUESTION 8

25 pts

8.1 a 6 / 6

✓ **- 0 pts** Correct

8.2 b 10 / 10

✓ **- 0 pts** Correct

8.3 c 6 / 6

✓ **- 0 pts** Correct

8.4 d 3 / 3

✓ **- 0 pts** Correct

QUESTION 9

25 pts

9.1 a 5 / 5

✓ **+ 5 pts** Correct

+ **1 pts** Method 1 (Polar Coordinates): Correctly substituted x = rcos(theta) and y = rsin(theta)

+ **1 pts** Method 1 (Polar Coordinates): Correct notation for the limit in r

+ **1 pts** Method 1 (Polar Coordinates): Correct lower bound for Squeeze Theorem

+ **1 pts** Method 1 (Polar Coordinates): Correct upper bound for Squeeze Theorem

+ **2 pts** Method 2 (Continuity): Split x^2 - y^2 into (|x| + |y|)(|x| - |y|)

+ **1 pts** Method 2 (Continuity): Reduce to |x| - |y| (or x - y if absolute values were dropped above)

+ **1 pts** Method 2 (Continuity): Applied continuity

+ **2 pts** Method 3 (Squeeze Theorem): Correct Upper Bound

+ **1 pts** Method 3 (Squeeze Theorem): Upper Bound has sign error or needs a further bound

+ **2 pts** Method 3 (Squeeze Theorem): Correct Lower Bound

+ **1 pts** Method 3 (Squeeze Theorem): Lower Bound has sign error or needs a further bound

+ **0 pts** Incorrect Method: Computing limit along different lines/curves

+ **1 pts** All Methods: Correctly identify that the limit is 0

+ **0 pts** No Credit

9.2 b 0 / 10

+ **10 pts** Correct

+ **2 pts** Method 1 (Polar Coordinates): Correctly substituted x = rcos(theta) and y = rsin(theta)

+ **2 pts** Method 1 (Polar Coordinates): Correct notation for the limit in r

+ **2 pts** Method 1 (Polar Coordinates): Correct lower bound for Squeeze Theorem

+ **2 pts** Method 1 (Polar Coordinates): Correct upper bound for Squeeze Theorem

+ **3 pts** Method 2 (Squeeze Theorem): Correct upper bound

+ **1.5 pts** Method 2 (Squeeze Theorem): Upper bound has a sign error (for example, dropping absolute value)

+ **3 pts** Method 2 (Squeeze Theorem): Correct lower bound

+ **1.5 pts** Method 2 (Squeeze Theorem): Lower bound has a sign error (for example, dropping absolute value)

+ **2 pts** Method 2 (Squeeze Theorem): Used Squeeze Theorem to justify the limit

✓ **+ 0 pts** **Incorrect Method:**

Computing limit along different lines/curves (Note that the limit does exist for this one)

+ **2 pts** All Methods:

Correctly identify that the limit is 0

+ **0 pts** No Credit

9.3 C **6 / 10**

+ **10 pts** Correct
+ **2 pts** 1st Limit: Valid path chosen

✓ + **2 pts** **1st Limit: Found limit along this path**

+ **2 pts** 2nd Limit: Valid path chosen

✓ + **2 pts** **2nd Limit: Found limit along this path**

✓ + **2 pts** **Deduce from different values that limit does not exist**

+ **0 pts** No credit and/or blank

You need to clearly indicate which paths you are taking these limits on

QUESTION 10

10 **20 / 20**

✓ - **0 pts** **Correct: Letting tangent plane be perpendicular to $$\langle -1,1,0\rangle$$ and contain $$(6,0,0)$$ (or a different point on $$\ell$$) and solving.**

Problem 1: Check this box if you used more than one page for this problem: ☑

(a) $$\frac{\partial}{\partial y}\left[ye^{x^2y^2}+\frac{1}{\sqrt{x+y}}+\ln\left(\ln\left(25+\sin(e^x)\right)\right)\right] = \boxed{2y^2e^{x^2y^2}+e^{x^2y^2}-\frac{1}{2(x+y)^{\frac{3}{2}}}}$$

$$=\underbrace{\frac{\partial}{\partial y}\left[ye^{x^2y^2}\right]}+\underbrace{\frac{\partial}{\partial y}\left[(x+y)^{-\frac{1}{2}}\right]}+\underbrace{\frac{\partial}{\partial y}\left[\ln\left(\ln(25+\sin(e^x))\right)\right]}$$

$$2y^2e^{x^2y^2}+e^{x^2y^2} - \frac{1}{2(x+y)^{\frac{3}{2}}} + 0$$

no y-term, thus $= 0$

(b) $$\frac{\partial^3}{\partial x\partial y\partial z}\left[e^{xyz}+x^3y^4z^5+\frac{\tan(\ln(y+z^4e^y))}{\sqrt{y^7+z^{14}}}\right] = \boxed{e^{xyz}(1+3xy+x^2y^2z^2)+60x^2y^3z^4}$$

$$=\underbrace{\frac{\partial^3}{\partial x\partial y\partial z}\left[e^{xyz}\right]}+\underbrace{\frac{\partial^3}{\partial x\partial y\partial z}\left[x^3y^4z^5\right]}+\underbrace{\frac{\partial^3}{\partial x\partial y\partial z}\left[\frac{\tan(\ln(y+z^4e^y))}{\sqrt{y^7+z^{14}}}\right]}$$

$$=\frac{\partial^3}{\partial x\partial y\partial z}\left[e^{xyz}\right]$$
$$=\frac{\partial^2}{\partial x\partial y}\left[xye^{xyz}\right]$$
$$=\frac{\partial}{\partial x}\left[x^2yze^{xyz}+xe^{xyz}\right]$$
$$=\frac{\partial}{\partial x}\left[e^{xyz}(x^2yz+x)\right]$$
$$=e^{xyz}(2xyz+1)+yze^{xyz}(x^2yz+x)$$
$$=e^{xyz}(1+3xyz+x^2y^2z^2)$$

$$=\frac{\partial^3}{\partial x\partial y\partial z}\left[x^3y^4z^5\right]$$
$$=\frac{\partial^2}{\partial x\partial y}\left[5x^3y^4z^4\right]$$
$$=\frac{\partial}{\partial x}\left[20x^3y^3z^4\right]$$
$$=60x^2y^3z^4$$

$$=\frac{\partial^3}{\partial y\partial z\partial x}\left[\frac{\tan(\ln(y+z^4e^y))}{\sqrt{y^7+z^{14}}}\right]$$ (by Clairaut's Theorem)
$$=\frac{\partial^2}{\partial y\partial z}\left[0\right]$$
$$=\frac{\partial}{\partial y}\left[0\right]$$
$$=0$$

$$e^{xyz}(1+3xy+x^2y^2z^2)+60x^2y^3z^4$$

Use this page if you need more room for one of your solutions.

Extra page for Problem $\boxed{1}$

$\sqrt{(1)^2+(-1)^2+(-2)^2} = \sqrt{6}$

(c) $\mathbf{V}$ is the vector whose direction and length allows for a movement from $(-1,1,2)$ to the origin

$V = \langle 1, -1, -2 \rangle$, $\|v\| = \sqrt{6}$

$e_v = u = \left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right\rangle$

$f(x,y,z) = \frac{x^2 z}{y^3}$

$\nabla f(x,y,z) = \left\langle \frac{2xz}{y^3}, -\frac{3x^2 z}{y^4}, \frac{x^2}{y^3} \right\rangle$

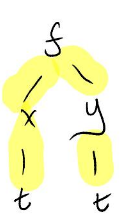
$\nabla f(-1,1,2) = \langle -4, -6, 1 \rangle$

$D_u f(-1,1,2) = u \cdot \nabla f(-1,1,2)$

$D_u f(-1,1,2) = \left\langle \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, -\frac{2\sqrt{6}}{6} \right\rangle \cdot \langle -4, -6, 1 \rangle$

$= -\frac{2\sqrt{6}}{3} + \frac{3\sqrt{6}}{3} - \frac{\sqrt{6}}{3} = \boxed{0}$

(d)



$g'(t) = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

$g'(t) = f_t = f_x \cdot x'(t) + f_y \cdot y'(t)$

$x'(t) = 12t^2 - 2t \Rightarrow x'(1) = 10$ $\quad x(1) = 3$

$y'(t) = \frac{5}{t} + 4\sin(t-1) \Rightarrow y'(1) = 5$ $\quad y(1) = -4$

$f_x = \frac{x}{\sqrt{x^2+y^2}} \Rightarrow f_x(3,4) = \frac{3}{5}$

$f_y = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow f_y(3,4) = \frac{4}{5}$

$\boxed{g'(1) = \frac{3}{5} \cdot 10 + \frac{4}{5} \cdot 5 = 6 + 4 = 10}$

Problem 2: Check this box if you used more than one page for this problem: ☐

(a) $r(t) = \langle 2e^{t}+4,\ 5-t,\ e^{2t} \rangle$

$\boxed{\text{arc length from } r(-1) \text{ to } r(2) \quad l = e^{4} - e^{-2} + 3}$

$r'(t) = \langle 2e^{t}, -1, 2e^{2t} \rangle$

$\|r'(t)\| = \sqrt{(2e^{t})^2 + (-1)^2 + (2e^{2t})^2} = \sqrt{4e^{2t} + 1 + 4e^{4t}}$

$(2e^{2t}+1)(2e^{2t}+1)$

$4e^{4t} + 2e^{2t} + 2e^{2t} + 1$

$= \sqrt{(2e^{2t}+1)^2}$

$= 2e^{2t}+1$

arc length formula $\boxed{\int_{t_1}^{t_2} \|r'(t)\|\, dt}$

$l = \int_{-1}^{2} 2e^{2t} + 1\, dt$

$= e^{2t} + t \Big|_{-1}^{2}$

$= (e^{4}+2) - (e^{-2}-1) = e^{4} - e^{-2} + 3$

(b)

curvature formula $\boxed{K = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}}$

$\begin{cases} 2e^{t}+4 = 6 \\ 5-t = 5 \\ e^{2t} = 1 \end{cases}$ occurs when $t = 0$

from above,

$r'(t) = \langle 2e^{t}, -1, 2e^{2t} \rangle, \quad r'(0) = \langle 2, -1, 2 \rangle, \quad \|r'(0)\| = 3$

$\sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$

then,

$r''(t) = \langle 2e^{t}, 0, 4e^{2t} \rangle, \quad r''(0) = \langle 2, 0, 4 \rangle$

$r'(0) \times r''(0) = \langle -4-0, 4-8, 0+2 \rangle = \langle -4, -4, 2 \rangle$

$\|r'(0) \times r''(0)\| = \sqrt{(-4)^2 + (-4)^2 + 2^2} = \sqrt{36} = 6$

$K(0) = \dfrac{\|r'(0) \times r''(0)\|}{\|r'(0)\|^3} = \dfrac{6}{3^3} = \dfrac{6}{27} = \boxed{\dfrac{2}{9}}$

Problem 3: Check this box if you used more than one page for this problem: ✓

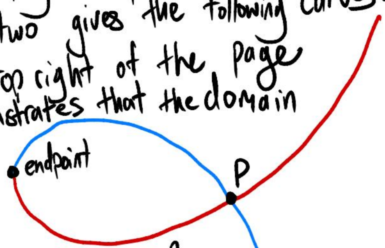
(a) $x = 3t^2 + 2$

$x - 2 = 3t^2$

$\frac{x-2}{3} = t^2$

$\pm\sqrt{\frac{x-2}{3}} = t$

splitting this parametric equation into two gives the following curves. the top right of the page demonstrates that the domain



is $x \geq 2$. These 2 curves intersect at two points: a shared end point & P.

By inspection, the endpoint MUST be at $x=2$ because the domain's edge has only a meeting of the two curves, NOT an intersection.

P must have an x value >2 because it is in the positive x direction in comparison to the end point.

parametric $x = 3t^2+2$ AND $y = 2t^3 - 2t + 3$

standard $y = 2\left(\sqrt{\frac{x-2}{3}}\right)^3 - 2\sqrt{\frac{x-2}{3}} + 3$ AND $y = 2\left(-\sqrt{\frac{x-2}{3}}\right)^3 + 2\sqrt{\frac{x-2}{3}} + 3$

defined for $x \geq 2$ (b/c of the $\sqrt{\frac{x-2}{3}}$ term) — defined for $x \geq 2$ (b/c of the $\sqrt{\frac{x-2}{3}}$ term)

$$2\left(\sqrt{\frac{x-2}{3}}\right)^3 - 2\sqrt{\frac{x-2}{3}} + 3 = 2\left(-\sqrt{\frac{x-2}{3}}\right)^3 + 2\sqrt{\frac{x-2}{3}} + 3$$

$$2\left(\sqrt{\frac{x-2}{3}}\right)^3 - 2\sqrt{\frac{x-2}{3}} = 2\left(-\sqrt{\frac{x-2}{3}}\right)^3 + 2\sqrt{\frac{x-2}{3}}$$

$$-2\sqrt{\frac{x-2}{3}} \qquad -2\sqrt{\frac{x-2}{3}}$$

$$2\left(\sqrt{\frac{x-2}{3}}\right)^3 - 4\sqrt{\frac{x-2}{3}} = -2\left(\sqrt{\frac{x-2}{3}}\right)^3$$

$$+2\left(\sqrt{\frac{x-2}{3}}\right)^3 \qquad +2\left(\sqrt{\frac{x-2}{3}}\right)^3$$

$$4\left(\sqrt{\frac{x-2}{3}}\right)^3 - 4\sqrt{\frac{x-2}{3}} = 0$$

$$4\sqrt{\frac{x-2}{3}}\left(\frac{x-2}{3} - 1\right) = 0$$

zero factor principle

$$\boxed{P = (5,3)}$$

$$4\sqrt{\frac{x-2}{3}} = 0$$

$$\sqrt{\frac{x-2}{3}} = 0$$

$$x - 2 = 0$$

$$x = 2$$

AND

$$\frac{x-2}{3} - 1 = 0$$

$$\frac{x-2}{3} = 1$$

$$x - 2 = 3$$

$$x = 5$$

$x = 2$ is the endpoint at the edge of the domain, so P must be at $\boxed{x = 5.}$

$$y = 2\left(\sqrt{\frac{x-2}{3}}\right)^3 - 2\sqrt{\frac{x-2}{3}} + 3$$

$$y = 2\left(\sqrt{\frac{5-2}{3}}\right)^3 - 2\sqrt{\frac{5-2}{3}} + 3$$

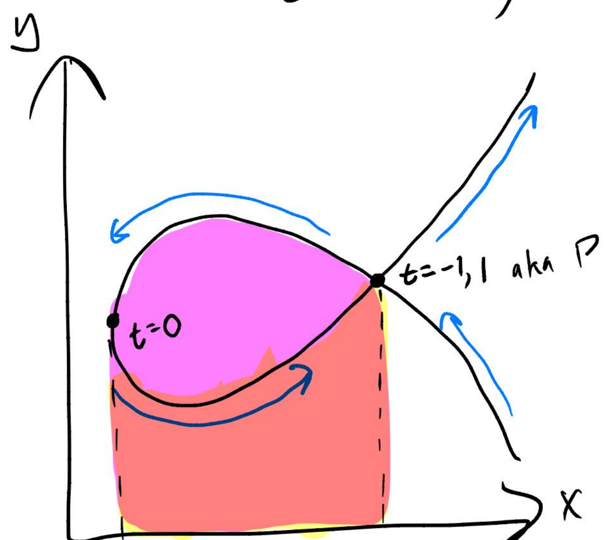
$$y = 2 - 2 + 3 = \boxed{3}$$

Use this page if you need more room for one of your solutions.

Extra page for Problem 3

(b) from previous...

$\pm\sqrt{\frac{x-2}{3}} = t$, @ $x=5, y=3 \rightarrow t=-1$ OR $t=1$



$y(-2) = -16+4+3 = -9$

$y(2) = 16-4+3 = 15$

thus, generally, as t increases, so does C

parametric equation area

$\int_{t_1}^{t_2} y(t)x'(t)dt$

thus, we want to integrate from $t=-1$ to $t=0$ (and take its absolute value b/c it's going in the negative-x direction) to find the area from the top of the region to the x-axis (the purple + red section) and then subtract the integral from $t=0$ to $t=1$ (the red section) so we're left with just the desired region which floats above the x axis

$\int_{-1}^{0} (2t^3-2t+3)(6t)\,dt$

$\int_{-1}^{0} 12t^4-12t^2+18t\,dt$

$\frac{12}{5}t^5-4t^3+9t^2\Big|_{-1}^{0}$

$\left(\frac{12}{5}(0)^5-4(0)^3+9(0)^2\right)-\left(\frac{12}{5}(-1)^5-4(-1)^3+9(-1)^2\right)$

$0+\frac{12}{5}-\frac{20}{5}-\frac{45}{5} = -\frac{53}{5}$

thus $\left|\int_{-1}^{0}(2t^3-2t+3)(6t)\,dt\right| = \frac{53}{5}$

$\int_{0}^{1} (2t^3-2t+3)(6t)\,dt$

$\int_{0}^{1} 12t^4-12t^2+18t\,dt$

$\frac{12}{5}t^5-4t^3+9t^2\Big|_{0}^{1}$

$\left(\frac{12}{5}(1)^5-4(1)^3+9(1)^2\right)-\left(\frac{12}{5}(0)^5-4(0)^3+9(0)^2\right)$

$\frac{12}{5}-\frac{20}{5}+\frac{45}{5} = \frac{37}{5}$

3.2
5) 16
-15
10

Area in region R

$$\boxed{\left|\int_{-1}^{0}(2t^3-2t+3)(6t)\,dt\right| - \int_{0}^{1}(2t^3-2t+3)(6t)\,dt = \frac{53}{5}-\frac{37}{5} = \frac{16}{5} = 3.2}$$

Problem 4: Check this box if you used more than one page for this problem: ☐

$f_x(x,y) = 3x^2 + 8xy^3$

$\int 3x^2 + 8xy^3 \, dx = x^3 + 4x^2y^3 + c(t) + C$

$f_y(x,y) = 12x^2y^2 + e^y$

$\int 12x^2y^2 + e^y \, dy = 4x^2y^3 + e^y + g(t) + C$

thus, $g(t) = x^3$

$c(t) = e^y$

$f(x,y) = 4x^2y^3 + x^3 + e^y + C$

$4 = 4 \cdot 0^2 \cdot 0^3 + 0^3 + e^0 + C = 1 + C$

$C = 3$

$\boxed{f(x,y) = 4x^2y^3 + x^3 + e^y + 3}$

Problem 5: Check this box if you used more than one page for this problem: ☐

$f(x,y) = (x^2+2xy)e^{-y}$

$2e^{-y}x + 2e^{-y}y$

$f_x = (2x+2y)e^{-y} = 2e^{-y}(x+y)$

$f_y = -(x^2+2xy)e^{-y} + (2x)e^{-y} = -xe^{-y}(x+2y-2)$

$f_{xx} = 2e^{-y}$

$f_{yy} = -2xe^{-y} + xe^{-y}(x+2y-2)$

$f_{xy} = -2e^{-y}x - 2e^{-y} = -2e^{-y}(x+1)$

$$\begin{cases} 2e^{-y}(x+y) = 0 \\ -xe^{-y}(x+2y-2) = 0 \end{cases}$$

$2e^{-y} = 0$, DNE

$x+y=0$, $y=-x$

$-xe^{x}(-x-2) = 0$

$-xe^{x} = 0$ $y=-x$

$x=0,\ y=0$ critical point #1

$(-x-2) = 0$

$x=-2,\ y=2$ critical point #2

$f_{xx}(0,0) = 2$ positive

$f_{yy}(0,0) = 0$ zero

$f_{xy}(0,0) = -2$ negative

$f_{xx}(-2,2) = 2e^{-2}$ positive

$f_{yy}(-2,2) = 4e^{-2}$ positive

$f_{xy}(-2,2) = -2e^{-2}(-1) = 2e^{-2}$ positive

for $x=0, y=0$:

$D = f_{xx}(0,0) * f_{yy}(0,0) - (f_{xy}(0,0))^2 = 0 - (-2)^2 = -4$ negative

thus, critical point $(x,y) = (0,0)$ is a saddle point by second derivative test

for $x=-2, y=2$:

$D = f_{xx}(-2,2) * f_{yy}(-2,2) - (f_{xy}(-2,2))^2 = 2e^{-2} \cdot 4e^{-2} - (2e^{-2})^2$

$f_{xx}(-2,2) = 2e^{-2}$ positive

$= 8e^{-4} - 4e^{-4} = 4e^{-4}$ positive

thus, critical point $(x,y) = (-2,2)$ is a local minimum by the second derivative test

Problem 6: Check this box if you used more than one page for this problem: ☐

First, find critical points.

$f_x = -y + 6$

$f_y = 2y - x - 2$

$$\begin{cases} -y + 6 = 0 \\ 2y - x - 2 = 0 \end{cases}$$

$y = 6$

$2 \cdot 6 - x - 2 = 0$

$10 - x = 0$

$x = 10$

one critical point:

$x = 10,\ y = 6$

HOWEVER, this point is not within $x \geq 0$, $x \leq y \leq 5$, so it is discarded.

Parameterize boundaries and find critical points/endpoints.

Top edge is $y = 5$.

$g(x) = f(x,5) = 25 - 5x + 6x - 10 = x + 15$ on $0 \leq x \leq 5$

$g'(x) = 1$

$g'(x) = 0$ at some x? No, x DNE for this case

so we'll check only the endpoints $(0,5)$ and $(5,5)$

Left edge is $x = 0$.

$h(y) = f(0,y) = y^2 - 0y + 6 \cdot 0 - 2y = y^2 - 2y$

$h'(y) = 2y - 2$

$h'(y) = 0$ at some y? Yes, at $y = 1$! Thus, we'll check $(0,1)$.

We'll also check the endpoints $(0,0)$ and $(0,5)$ (already found)

Right (diagonal) edge is $y = x$.

$q(x,x) = q(x) = x^2 - x^2 + 6x - 2x = 4x$

$q'(x) = 4$

$q'(x) = 0$ at some x? No, x DNE for this case

So we'll check only $(0,0)$ and $(5,5)$, which were both already found.

Thus, we will check:

- $(0,5)$
- $(0,1)$
- $(5,5)$
- $(0,0)$

For $(0,5)$:

$f(0,5) = 5^2 - 0 \cdot 5 + 6 \cdot 0 - 2 \cdot 5$
$= 25 - 10 = 15$

For $(5,5)$:

$f(5,5) = 5^2 - 5 \cdot 5 + 6 \cdot 5 - 2 \cdot 5$
$= 30 - 10 = 20$ largest

For $(0,0)$:

$f(0,0) = 0^2 - 0 \cdot 0 + 6 \cdot 0 - 2 \cdot 0$
$= 0$

For $(0,1)$:

$f(0,1) = 1^2 - 0 \cdot 1 + 6 \cdot 0 - 2 \cdot 1$
$= -1$ smallest

Thus, the global maximum which f takes in region T is $\boxed{20}$ and occurs at $\boxed{(5,5)}$ and the global minimum which f takes in region T is $\boxed{-1}$ and occurs at $\boxed{(0,1)}$.

$$\begin{tikzcd}[column sep=small] & f \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

Problem 7: Check this box if you used more than one page for this problem: ☐

$$g'(t) = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$g'(t) = f_t = f_x \cdot x'(t) + f_y \cdot y'(t)$$

$$x(t) = \ln(1) - \sin(0) = 0$$

$$y(t) = 1^5 - 1^5 - 1^5 = -1$$

thus $x = 0, \ y = -1$

$$x'(t) = \frac{1}{t} + \cos(1-t) \rightarrow x'(1) = 1 + \cos(0) = 2$$

$$y'(t) = 5t^4 - 3t^2 - 1 \rightarrow y'(1) = 5 - 3 - 1 = 1$$

$$g'(1) = F_t(1) = f_x(0,-1) \cdot x'(1) + f_y(0,-1) \cdot y'(1)$$

$$= -6 \cdot 2 + 7 \cdot 1 = 7 - 12 = -5$$

$$\boxed{g'(1) = -5}$$

Problem 8: Check this box if you used more than one page for this problem: ✓

(a) plugging in $x=3$ & $y=0$

$$z^7 + 3^2 \cdot 0^2 \cdot z^4 + 3 \cdot e^0 \cdot z = 4$$

$$z^7 + 3z = 4$$

$$z(z^6+3) = 4$$

by inspection, if $z=1$, $1(1^6+3)=4$ ✓

positive positive negative

$$\boxed{+z^7}\boxed{+3z}\boxed{-4} = 0$$

change in sign

← a real root of z^7+3z-4 is an answer

because there is a single change in the sign of a term when moving from the left to the right of the polynomial (when it's ordered from highest to lowest order), there's one positive real root, which is the answer $z=1$ we found.

(complex roots come in pairs, so this number cannot be reduced)

$$(-z)^7 - 3z - 4 = 0$$

$$\boxed{-z^7}\boxed{-3z}\boxed{-4} = 0$$

negative negative negative

when z is replaced w/ $-z$, there are no changes in sign. Therefore, there are 0 negative roots.

0 is not a solution to $z^7+3z-4=0$ as $-4 \neq 0$

There fore, $z=1$ is the only answer.

(b) For $\frac{\partial z}{\partial x}$:

$$\frac{\partial}{\partial x}\left(z^7 + x^2yz^4 + xe^yz = 4\right)$$

$$7z^6\frac{\partial z}{\partial x} + x^2y\cdot 4z^3\frac{\partial z}{\partial x} + 2xyz^4 + xe^y\frac{\partial z}{\partial x} + e^yz = 0$$

$$7z^6\frac{\partial z}{\partial x} + 4x^2yz^3\frac{\partial z}{\partial x} + xe^y\frac{\partial z}{\partial x} = -2xyz^4 - e^yz$$

$$\frac{\partial z}{\partial x}\left(7z^6 + 4x^2yz^3 + xe^y\right) = -2xyz^4 - e^yz$$

$$f_x = \frac{\partial z}{\partial x} = \frac{-(2xyz^4+e^yz)}{7z^6+4x^2yz^3+xe^y} = -\frac{2xyz^4+e^yz}{7z^6+4x^2yz^3+e^yx}$$

$z=f(3,0)=1$

$$f_x(3,0) = -\frac{2\cdot3\cdot0\cdot1^4+e^0\cdot1}{7\cdot1^6+4\cdot3^2\cdot0\cdot1^3+e^0\cdot3} = -\frac{1}{7+3} = -\frac{1}{10}$$

$$\frac{\partial}{\partial y}\left(z^7 + x^2yz^4 + xe^yz = 4\right)$$

$$7z^6\cdot\frac{\partial z}{\partial y} + x^2y\cdot4z^3\frac{\partial z}{\partial y} + x^2z^4 + e^yx\cdot\frac{\partial z}{\partial y} + xe^yz = 4$$

$$7z^6\cdot\frac{\partial z}{\partial y} + 4x^2yz^3\cdot\frac{\partial z}{\partial y} + e^yx\cdot\frac{\partial z}{\partial y} = -x^2z^4 - xe^yz$$

$$\frac{\partial z}{\partial y}\left(7z^6+4x^2yz^3+e^yx\right) = -\left(x^2z^4+xe^yz\right)$$

$$f_y = \frac{\partial z}{\partial y} = \frac{-(x^2z^4+xe^yz)}{7z^6+4x^2yz^3+e^yx} = -\frac{x^2z^4+xe^yz}{7z^6+4x^2yz^3+e^yx}$$

$$f_y(3,0) = -\frac{3^2\cdot1^4+e^0\cdot1\cdot3}{7\cdot1^6+4\cdot3^2\cdot0\cdot1^3+e^0\cdot3} = -\frac{12}{7+3} = -\frac{12}{10} = -\frac{6}{5}$$

$$\nabla f = \langle f_x, f_y\rangle$$

$$\nabla f(3,0) = \langle f_x(3,0), f_y(3,0)\rangle = \boxed{\left\langle -\frac{1}{10}, -\frac{6}{5}\right\rangle}$$

Use this page if you need more room for one of your solutions.

Extra page for Problem 8

(c) $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

from previous...
$f(3,0) = 1$
$f_x(3,0) = -\frac{1}{10}$
$f_y(3,0) = -\frac{6}{5}$

$L(x,y) = f(3,0) + f_x(3,0)(x-3) + f_y(3,0)(y-0)$
for (3,0)

$L(x,y) = 1 - 0.1(x-3) - 1.2y$

$L(x,y) = 1 - 0.1x + 0.3 - 1.2y$

$\boxed{L(x,y) = 1.3 - 0.1x - 1.2y}$

(d)

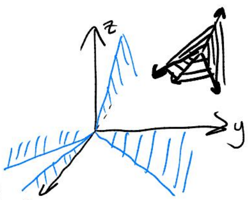
$$\begin{array}{r} .298 \\ .012 \\ \hline .310 \end{array}$$

$L(2.98, 0.01) = 1.3 - 0.1 \cdot 2.98 - 1.2 \cdot 0.01$
$= 1.3 - 0.298 - 0.012$
$= 1.3 - 0.310$
$= 0.99$

$\boxed{f(2.98, 0.01) \approx 0.99}$

Problem 9: Check this box if you used more than one page for this problem: ☐

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{|x|+|y|}$



$$-|y| \le \frac{x^2-y^2}{|x|+|y|} \le |x|$$

$$\lim_{(x,y)\to(0,0)}\left(-|y| \le \frac{x^2-y^2}{|x|+|y|} \le |x|\right)$$

$$\lim_{(x,y)\to(0,0)} -|y| \le \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{|x|+|y|} \le \lim_{(x,y)\to(0,0)} |x|$$

$$-|0| \le \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{|x|+|y|} \le |0|$$

$$\underbrace{0 \le \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{|x|+|y|} \le 0}_{=0}$$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{|x|+|y|} = 0}$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^6+y^4}$ $\boxed{\text{The limit does not exist.}}$

$$\lim_{(x,0)\to(0,0)} \frac{x\cdot 0^4}{x^6+0^4} = \lim_{x\to 0} \frac{x\cdot 0^4}{x^6+0^4} = \frac{0}{x^6} = 0$$

$$\lim_{(x,x^2)\to(0,0)} \frac{x\cdot(x^2)^4}{x^6+(x^2)^4} = \lim_{x\to 0} \frac{x^9}{x^6+x^8} = \lim_{x\to 0} \frac{x^9}{x^6(1+x^8)} = \lim_{x\to 0} \frac{x^3}{1+x^8} = \frac{0^3}{1+0} = 0$$

$$\lim_{(y^4,y)\to(0,0)} \frac{y^4y^4}{(y^4)^6+y^4} = \lim_{y\to 0} \frac{y^8}{y^{24}+y^4} = \lim_{y=0} \frac{y^8}{y^4(y^{20}+1)} = \lim_{y\to 0} \frac{y^4}{y^{20}+1} = \frac{0}{0+1} = 0$$

$$\lim_{(x^4,x^6)\to(0,0)} \frac{x^4x^{24}}{x^{24}+x^{24}} = \lim_{x\to 0} \frac{x^4x^{24}}{2x^{24}} = \lim_{x\to 0} \frac{x^{28}}{2x^{24}} = \lim_{x\to 0} \frac{x^4}{2} = 0$$

$$\lim_{r\to 0} \frac{r^5\cos\theta\sin^4\theta}{r^4(r^2\cos^6\theta+\sin^4\theta)} = \lim_{r\to 0} \frac{r\cos\theta\sin^4\theta}{r^2\cos^6\theta+\sin^4\theta} = 0$$

$$\lim_{(x,e^x)\to(0,0)} \frac{xe^{4x}}{x^6+e^{4x}} = \lim_{x\to 0} \frac{xe^{4x}}{x^6+e^{4x}} = \frac{0\cdot 1}{0+1} = 0$$

$$\lim_{(e^x,e^y)\to(0,0)} \frac{e^xe^{4y}}{e^{6x}+e^{4y}} = \frac{e^0}{e^0+e^0} = \frac{1}{1+1} = \frac{1}{2} \ne 0$$

$$\lim_{(x^{\frac{1}{6}},x^{\frac{1}{4}})\to(0,0)} \frac{\sqrt[6]{x}\sqrt[4]{x}}{2x} = \lim_{x\to 0} \frac{x^{\frac{10}{24}}}{2x} = \lim_{x\to 0} \frac{1}{2x^{\frac{14}{24}}} = 0$$

$$\lim_{(x,x^{\frac{1}{2}})\to(0,0)} \frac{x\,x^2}{x^6+x^2} = \lim_{x\to 0} \frac{x^3}{x^2(x^6+1)} = \lim_{x\to 0} \frac{x}{x^6+1} = 0$$

(c)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+xz+yz}{x^2+y^2+z^2}$$

$$\lim_{(x,x,x)\to(0,0,0)} \frac{xx+xx+xx}{x^2+x^2+x^2} = \lim_{x\to 0} \frac{x^2+x^2+x^2}{x^2+x^2+x^2} = 1$$

$$\lim_{(0,y,0)\to(0,0,0)} \frac{0\cdot y+0\cdot 0+y\cdot 0}{y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

$\boxed{\text{Thus, this limit does not exist.}}$

Problem 10: Check this box if you used more than one page for this problem: ☑

$2x^2+y^2+z=18$

has to be on the surface

$f(x,y,z)=2x^2+y^2+z-18$

$\nabla f(x,y,z)=\langle 4x, 2y, 1\rangle$

$\langle 4x,2y,1\rangle\cdot(\langle 6,0,0\rangle-\langle x,y,z\rangle)=0$

$\langle 4x,2y,1\rangle\cdot\langle 6-x,-y,-z\rangle=0$

$24x-4x^2-2y^2-z=0$

has to contain the line

$\langle -1,1,0\rangle\cdot\langle 4x,2y,1\rangle=0$

$-4x+2y=0$

$2x=y$

has to be parallel to the line

$$\begin{cases} 24x-4x^2-2y^2-z=0 \rightarrow 24x-4x^2-8x^2-18+6x^2=0 \\ 2x^2+y^2+z=18 \rightarrow 2x^2+4x^2+z=18 \\ \qquad\qquad\qquad\quad z=18-6x^2 \\ 2x=y \end{cases}$$

$\rightarrow -6x^2+24x-18=0$

$-6(x^2-4x-3)=0$

$6((x-3)(x-1))=0$

thus, $x=3$ & $x=1$

$2\cdot 3=6$

$18+36+z=18$

$\boxed{x=3 \quad y=6 \quad z=-36}$

$2+4+z=18$

$2\cdot 1=2$

$\boxed{x=1 \quad y=2 \quad z=12}$

Use this page if you need more room for one of your solutions.

Extra page for Problem 10

Point 1:

$(1,2,12)$ w/ normal $\langle 4,4,1\rangle$

$\langle 4,4,1\rangle \cdot \langle x-1, y-2, z-12\rangle = 0$

$4x-4+4y-8+z-12=0$

$P_1:\ 4x+4y+z=24$

Point 2:

$(3,6,-36)$

w/ normal

$\langle 12,12,1\rangle$

$\langle 12,12,1\rangle \cdot \langle x-3, y-6, z+36\rangle = 0$

$12x-36+12y-72+z+36=0$

$P_2:\ 12x+12y+z=72$

Final (Math 32A)

Tuesday, March 20th

You will have 24 hours to complete this exam. It will be due at 12:00 am on Saturday, 3/21. Please upload your solutions to CCLE before the deadline. You are strongly encouraged to use the solution template posted on CCLE for your solutions. If you do not use the template, make sure you still format your solutions in the same way (so name, ID and *signature* on page 1, Problem 1 on page 2, Problem 2 on page 3, and so on).

The exam is open book/open notes. You can use any materials including the notes, your homeworks and quizzes, and anything you can find on the internet. However you are not allowed to get help from any other person. This includes talking to someone in person, talking to someone online, or asking for help on any sort of online forum. Also it includes asking specific questions about the questions on the final, OR more general questions about the material in the course.

While you may use any resources you can find, your submitted solutions cannot rely on anything other than pencil and paper. That is, even if you used some advanced resources to get your answer originally, or to check that it is correct, in order for your solution to be considered correct, you must include all of the steps you would have taken to get the solution if you did not have access to a calculator or a computer.

Show your work for these problems, don't just give an answer. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may use any results from class, the textbook or the homework sets, but please make it clear when you are doing so. You may still earn partial credit even if your final answer is incorrect.

Question	Points	Score
1	30	
2	15	
3	20	
4	10	
5	20	
6	20	
7	15	
8	25	
9	25	
10	20	
Total:	200	

Name:__________________________________

1. [30 pts] Find the indicated derivatives:

(a) [5 pts] $\dfrac{\partial}{\partial y}\left[ye^{x^2y^2} + \dfrac{1}{\sqrt{x+y}} + \ln(\ln(25+\sin(e^x)))\right]$

(b) [8 pts] $\dfrac{\partial^3}{\partial x \partial y \partial z}\left[e^{xyz} + x^3y^4z^5 + \dfrac{\tan(\ln(y^2+z^4e^y))}{\sqrt{y^7+z^{14}}}\right]$

(c) [8 pts] The directional derivative $D_{\mathbf{u}}f(-1,1,2)$, where $f(x,y,z) = \dfrac{x^2z}{y^3}$ and $\mathbf{u}$ is the unit vector pointing from $(-1,1,2)$ to the origin.

(d) [9 pts] Let $f(x,y) = \sqrt{x^2+y^2}$, and let $g(t) = f\left(4t^3 - t^2,\, 5\ln(t) - 4\cos(t-1)\right)$. Find $g'(1)$.

Name:____________________________________

2. [15 pts] Let $\mathcal{C}$ be the curve defined by the vector valued function

$$\mathbf{r}(t) = \left\langle 2e^t + 4, 5 - t, e^{2t} \right\rangle$$

on the interval $-1 \leq t \leq 2$.

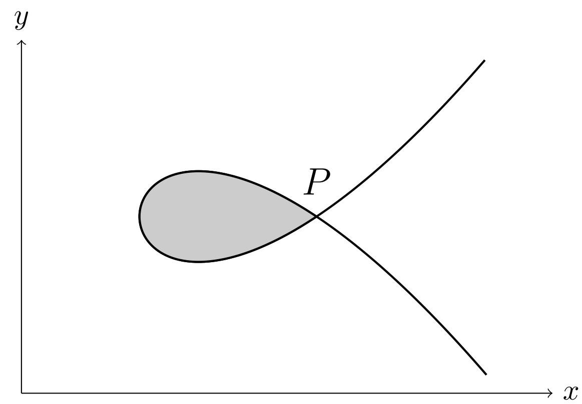
(a) [8 pts] Find the arc length of $\mathcal{C}$ (from the point $\mathbf{r}(-1)$ to the point $\mathbf{r}(2)$).

(b) [7 pts] Find the curvature of $\mathcal{C}$ at the point $(6, 5, 1)$.

Name:__________________________________

3. [20 pts] The curve $\mathcal{C}$ in the xy-plane is given by the parametric equations

$$x = 3t^2 + 2 \qquad\qquad y = 2t^3 - 2t + 3.$$



(a) [7 pts] Find the (x, y) coordinates of the point P, where the graph of $\mathcal{C}$ intersects itself.

(b) [13 pts] Find the area of the shaded region, enclosed by the graph of $\mathcal{C}$.

Name:___________________________________

4. [10 pts] $f(x,y)$ is a differentiable function satisfying

$$f(0,0) = 4, \qquad f_x(x,y) = 3x^2 + 8xy^3, \qquad \text{and} \qquad f_y(x,y) = 12x^2y^2 + e^y.$$

Find $f(x,y)$.

Name:______________________________

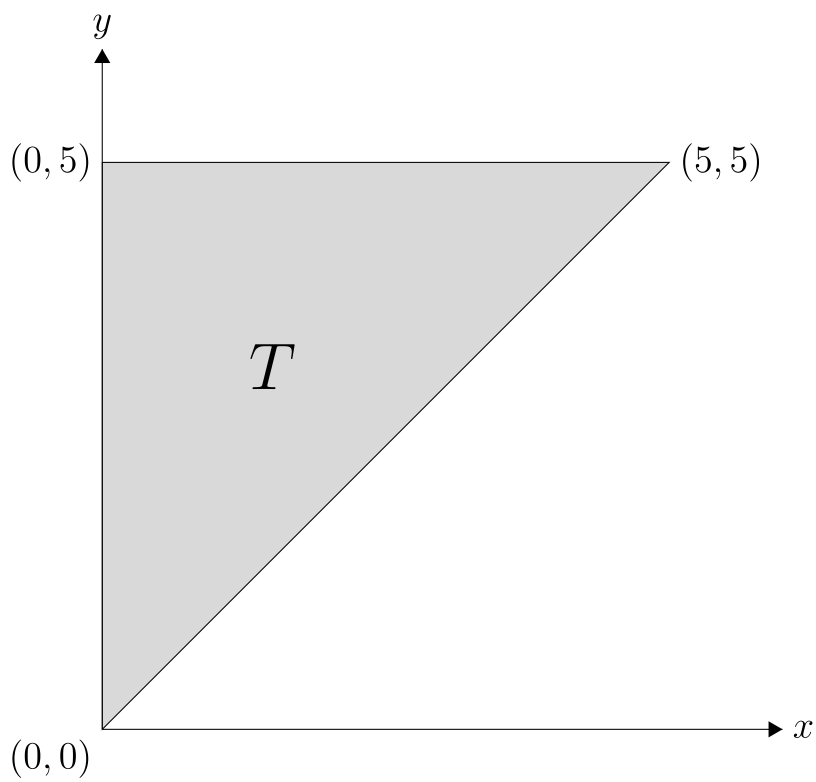
5. [20 pts] Find all critical point(s) of the function

$$f(x, y) = (x^2 + 2xy)e^{-y}.$$

Classify each one as a local maximum, a local minimum, or a saddle point.

Name:________________________________

6. [20 pts] Let $f(x, y) = y^2 - xy + 6x - 2y$. Find the (global) maximum and minimum values which f takes in the triangular region T, with vertices at $(0, 0)$, $(0, 5)$ and $(5, 5)$ (i.e. the region defined by the inequalities $x \geq 0$, and $x \leq y \leq 5$), and the points at which they occur.



Name:____________________________________

7. [15 pts] The function $f(x, y)$ is differentiable, but you do not have a formula for it. The following values of f and of its partial derivatives are known:

$f(-2,-2)=5$ $f_x(-2,-2)=4$ $f_y(-2,-2)=3$	$f(-2,-1)=-1$ $f_x(-2,-1)=-1$ $f_y(-2,-1)=-1$	$f(-2,0)=0$ $f_x(-2,0)=4$ $f_y(-2,0)=-5$	$f(-2,1)=10$ $f_x(-2,1)=-2$ $f_y(-2,1)=4$	$f(-2,2)=12$ $f_x(-2,2)=11$ $f_y(-2,2)=-3$
$f(-1,-2)=8$ $f_x(-1,-2)=2$ $f_y(-1,-2)=1$	$f(-1,-1)=11$ $f_x(-1,-1)=13$ $f_y(-1,-1)=17$	$f(-1,0)=-2$ $f_x(-1,0)=-7$ $f_y(-1,0)=3$	$f(-1,1)=6$ $f_x(-1,1)=9$ $f_y(-1,1)=1$	$f(-1,2)=9$ $f_x(-1,2)=0$ $f_y(-1,2)=-1$
$f(0,-2)=-8$ $f_x(0,-2)=-5$ $f_y(0,-2)=10$	$f(0,-1)=11$ $f_x(0,-1)=-6$ $f_y(0,-1)=7$	$f(0,0)=0$ $f_x(0,0)=0$ $f_y(0,0)=0$	$f(0,1)=1$ $f_x(0,1)=6$ $f_y(0,1)=3$	$f(0,2)=3$ $f_x(0,2)=3$ $f_y(0,2)=3$
$f(1,-2)=11$ $f_x(1,-2)=0$ $f_y(1,-2)=0$	$f(1,-1)=0$ $f_x(1,-1)=-3$ $f_y(1,-1)=-7$	$f(1,0)=3$ $f_x(1,0)=-6$ $f_y(1,0)=9$	$f(1,1)=4$ $f_x(1,1)=2$ $f_y(1,1)=3$	$f(1,2)=2$ $f_x(1,2)=-7$ $f_y(1,2)=5$
$f(2,-2)=3$ $f_x(2,-2)=1$ $f_y(2,-2)=8$	$f(2,-1)=6$ $f_x(2,-1)=9$ $f_y(2,-1)=1$	$f(2,0)=6$ $f_x(2,0)=-4$ $f_y(2,0)=-2$	$f(2,1)=-7$ $f_x(2,1)=5$ $f_y(2,1)=8$	$f(2,2)=2$ $f_x(2,2)=2$ $f_y(2,2)=2$

Let $g(t) = f\left(\ln(t) - \sin(1-t), t^5 - t^3 - t\right)$. Find $g'(1)$.

Be clear about which values from the table you are using!

Name:______________________________

8. [25 pts] The function $z = f(x, y)$ is defined implicitly by the equation

$$z^7 + x^2yz^4 + xe^yz = 4.$$

(that is, the point $(x, y, f(x, y))$ is a point on the surface $z^7 + x^2yz^4 + xe^yz = 4$ for every point (x, y) in the domain of f.)

(a) [6 pts] Find $f(3, 0)$. [*Hint: Trying to solve the equation directly for z may be too hard, but if you plug in $x = 3$ and $y = 0$ there should be an "obvious" value of z that works. Show that it's the only value of z that works.*]

(b) [10 pts] Find the gradient $\nabla f(3, 0)$.

(c) [6 pts] Find the *linearization* $L(x, y)$ for $f(x, y)$ at $(3, 0)$ (that is, a linear function $L(x, y)$ with $L(x, y) \approx f(x, y)$ for $(x, y) \approx (3, 0)$).

(d) [3 pts] Use the approximation from part (c) to estimate $f(2.98, 0.01)$.

Name:____________________________________

9. [25 pts] Compute the following limits, or show that they do not exist. Explain your reasoning, don't just give an answer with no explanation:

(a) [5 pts] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{|x| + |y|}$

(b) [10 pts] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^6 + y^4}$

(c) [10 pts] $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + xz + yz}{x^2 + y^2 + z^2}$

Name:________________________________

10. [20 pts] Let ℓ be the line defined by the vector valued function $\mathbf{r}(t) = \langle 6 - t, t, 0 \rangle$ and let $\mathcal{S}$ be the surface with equation

$$2x^2 + y^2 + z = 18$$

There are two distinct planes $\mathcal{P}_1$ and $\mathcal{P}_2$ that contain the line ℓ and are tangent to the surface $\mathcal{S}$. Find equations (involving just the variables x, y and z and no other variables) for the planes $\mathcal{P}_1$ and $\mathcal{P}_2$, and the points at which they are tangent to the surface $\mathcal{S}$.