Name: Bryan Luo

1. [25 pts] Find the indicated derivatives:

(a) [6 pts] $\dfrac{\partial}{\partial y}\left[\dfrac{\tan x}{2x+y} + \cos(x^3y^2) + \sqrt[3]{\dfrac{\cos x}{12+\sin^3 x}}\right].$

$$-\frac{\tan x}{(2x+y)^2} - \sin(x^3y^2)\,2y$$

(b) [9 pts] The directional derivative $D_{\mathbf{u}}f(-1,1,-5)$, where

$$f(x,y,z) = \frac{y^3}{x^2} - 3z$$

and $\mathbf{u} = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$ (note that $\mathbf{u}$ is a unit vector).

$$f_x(x,y,z) = -2\frac{y^3}{x^3}$$

$$f_z(x,y,z) = -3$$

$$f_y(x,y,z) = \frac{3}{x^2}y^2$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \left\langle -\frac{2y^3}{x^3}, \frac{3}{x^2}y^2, -3 \right\rangle$$

$$\nabla f(-1,1,-5) = \langle 2, 3, -3 \rangle$$

$$D_u f(-1,1,-5) = \nabla f(-1,1,-5) \cdot u$$

$$= \langle 2,3,-3 \rangle \cdot \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

$$= \frac{4}{3} - 1 - 2 = \frac{1}{3} - 2 = \frac{1}{3} - \frac{6}{3} = -\frac{5}{3}$$

$$D_u f(-1,1,-5) = -\frac{5}{3}$$

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(c) [10 pts] $\dfrac{\partial^2}{\partial x \partial y}\left[x\sin(xy) + \dfrac{1}{2x+5y} + \dfrac{y^4}{y^{3/2}+\cos(y^2+1)}\right]$

$$\frac{\partial}{\partial x}\left[x\sin(xy) + \frac{1}{2x+5y} + \frac{y^4}{y^{3/2}+\cos(y^2+1)}\right]$$

$$= \sin(xy) + xy\cos(xy) + (-1)(2x+5y)^{-2}$$

$$\frac{\partial}{\partial y}\ \sin(xy) + xy\cos(xy) - (2x+5y)^{-2}$$

$$= x\cos(xy) + x\cos(xy) + xy(-\sin(xy)x) + 2(2x+5y)^{-3}5$$

$$= 2x\cos(xy) - x^2y\sin(xy) + 10\frac{1}{(2x+5y)^3}$$

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2. [10 pts] Let $f(x,y) = (x^3 + y^2)^{2/3}$.

(a) [7 pts] Find the linearization $L(x,y)$ for $f(x,y)$ at the point $(x_0, y_0) = (-1,3)$ (that is, a linear function $L(x,y)$ with $L(x,y) \approx f(x,y)$ for $(x,y) \approx (-1,3)$).

$$f_x(x,y) = \frac{2}{3}(x^3+y^2)^{-1/3}\, 3x^2$$

$$f_y(x,y) = \frac{2}{3}(x^3+y^2)^{-1/3}\, 2y$$

$$f(-1,3) = (-1+9)^{2/3} = 8^{2/3} = 4$$

$$f_x(-1,3) = \frac{2}{3}(-1+9)^{-1/3}\, 3(-1)^2 = \frac{2}{3}(8)^{-1/3}\, 3 = 2(8)^{-1/3} = 2\frac{1}{8^{1/3}} = 1$$

$$f_y(-1,3) = \frac{2}{3}(-1+9)^{-1/3}\, 2(3) = \frac{2}{3}\frac{1}{(8)^{1/3}}\, 6 = 4\left(\frac{1}{2}\right) = 2$$

$$L(x,y) = f(-1,3) + f_x(-1,3)(x+1) + f_y(-1,3)(y-3)$$

$$L(x,y) = 4 + (x+1) + 2(y-3)$$

(b) [3 pts] Use the approximation from part (a) to estimate $f(-1.03, 2.98)$.

$$L(-1.03, 2.98) = 4 + (-0.03) + 2(-0.02)$$

$$= 4 - 0.03 - 0.04$$

$$= 4 - 0.07$$

$$= 3.93$$

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3. [20 pts] Let C be the curve defined by the vector valued function

$$\mathbf{r}(t) = \left\langle 4 - t, \frac{4t^{3/2} - 1}{3}, t^2 \right\rangle$$

on the interval $0 \le t \le 4$.

(a) [10 pts] Find the arc length of C (from the point $\mathbf{r}(0)$ to the point $\mathbf{r}(4)$).

$$r'(t) = \langle -1, 2t^{1/2}, 2t \rangle$$

$$s = \int_0^4 \|r'(t)\| \, dt = \int_0^4 \sqrt{1 + 4t + 4t^2} \, dt = \int_0^4 \sqrt{(2t+1)^2} \, dt$$

$$= \int_0^4 2t + 1 \, dt = (t^2 + t) \Big|_0^4 = 4^2 + 4 = 20$$

$$\boxed{20}$$

(b) [10 pts] Find the curvature of C at the point $(3, 1, 1)$.

$$r''(t) = \langle 0, t^{-1/2}, 2 \rangle$$

at $t = 1$

$$r'(1) = \langle -1, 2, 2 \rangle \qquad r''(1) = \langle 0, 1, 2 \rangle$$

$$\kappa(1) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$$r'(1) \times r''(1) = \langle -1, 2, 2 \rangle \times \langle 0, 1, 2 \rangle = 2i + 2j - 1k = \langle 2, 2, -1 \rangle$$

$$\kappa(1) = \frac{\sqrt{2^2 + 2^2 + (-1)^2}}{\left(\sqrt{(-1)^2 + (2)^2 + (2)^2}\right)^3} = \frac{3}{(3)^3} = \frac{3}{27} = \frac{1}{9}$$

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4. [25 pts] Let $f(x, y, z) = 2xy + xz + yz$.

(a) [2 pts] Find $\nabla f(x, y, z)$. $f_x(x,y,z) = 2y + z$ $f_z(x,y,z) = x + y$
$f_y(x,y,z) = 2x + z$

$\nabla f(x,y,z) = \langle 2y+z,\ 2x+z,\ x+y \rangle$

(b) [6 pts] Find the maximum possible rate of change of f at the point $P = (2, 1, 0)$, and the direction in which it occurs. (That is, find the largest possible value of $D_{\mathbf{u}}f(2, 1, 0)$, where $\mathbf{u}$ is a *unit* vector, and the unit vector $\mathbf{u}$ for which it occurs.)

$\nabla f(2,1,0) = \langle 2, 4, 3 \rangle$

max $D_u f(2,1,0)$ is $\|\nabla f(2,1,0)\| = \sqrt{2^2+4^2+3^2} = \sqrt{4+16+9}$
$= \sqrt{29}$

Direction
$= \frac{1}{\|\nabla f(2,1,0)\|} \nabla f(2,1,0) = \frac{1}{\sqrt{2^2+4^2+3^2}} \langle 2,4,3 \rangle = \frac{1}{\sqrt{29}} \langle 2,4,3 \rangle$

(c) [7 pts] Find a Cartesian equation (i.e. one relating x, y and z, with no additional variables) for the plane tangent to the surface $f(x, y, z) = 4$ at the point $(2, 1, 0)$.

normal vector $n = \langle f_x(2,1,0), f_y(2,1,0), f_z(2,1,0) \rangle$

$n = \langle 2, 4, 3 \rangle$

Point : $(2,1,0)$

Plane: $2(x-2) + 4(y-1) + 3(z) = 0$

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(d) [10 pts] Find all points on the surface

$$2xy + xz + yz = 6$$

(i.e. the surface $f(x, y, z) = 6$, where f is the function from the previous page) where the tangent plane is parallel to the plane $2x + 2y + z = -11$.

Normal vector of $K\langle 2,2,1\rangle$

Plane's normal vector:

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2y+z,\ 2x+z,\ x+y \rangle$$

$$\langle 2y+z,\ 2x+z,\ x+y \rangle = K\langle 2,2,1\rangle$$

$2y+z = 2K$ $2x+z=2K$ $x+y=K$

$2y+2y=2K$ $2(K-y)+z=2K$ $x=K-y$

$4y=2K$ $2K-2y+z=2K$

$y=\frac{1}{2}K$ $z=2K+2y-2K=2y$ $x=K-\frac{1}{2}K=\frac{1}{2}K$

$z=2(\frac{1}{2}K)=K$

$$K\left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

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5. [20 pts] Compute the following limits, or show that they do not exist. Explain your reasoning, don't just give an answer with no explanation:

(a) [10 pts] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{x^4+y^4}$.

$$\lim_{r\to 0} \frac{r^2\cos^2\theta\,\sin^2(r\sin\theta)}{r^4\cos^4\theta + r^4\sin^4\theta} =$$

$$= \lim_{r\to 0} \frac{\cos^2\theta\,\sin^2(\sin\theta)}{r(\cos^4\theta+\sin^4\theta)}$$

Does not exist because limit depends on the angle θ

(b) [10 pts] $\displaystyle\lim_{(x,y)\to(0,0)} (2x^2+3y^2)\sin^2\left(\frac{1}{x^4+y^6}\right)$.

$$\lim_{(x,y)\to(0,0)} -2x^2-3y^2 \le \lim_{(x,y)\to(0,0)} (2x^2+3y^2)\sin^2\left(\frac{1}{x^4+y^6}\right) \le \lim_{(x,y)\to(0,0)} 2x^2+3y^2$$

$$0 \le \lim_{(x,y)\to(0,0)} (2x^2+3y^2)\sin^2\left(\frac{1}{x^4+y^6}\right) \le 0$$

$$\lim_{(x,y)\to(0,0)} (2x^2+3y^2)\sin^2\left(\frac{1}{x^4+6}\right) = 0$$

~~by sq sq~~ by squeeze theorem

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