Name: Bryan Luo

1. [35 pts] Compute the following quantities:

(a) [5 pts] $\cos\theta$, where θ is the angle between $\mathbf{a} = \langle 1, -2, -2\rangle$ and $\mathbf{b} = \langle 2, 2, 1\rangle$.

$$\cos\theta = \frac{a \cdot b}{\|a\|\|b\|} = \frac{(1)(2)+(-2)(2)+(-2)(1)}{\sqrt{1^2+(-2)^2+(-2)^2}\sqrt{2^2+2^2+1^2}} = \frac{-4}{\sqrt{9}\sqrt{9}}$$

$$\cos\theta = \frac{-4}{(3)(3)} = \frac{-4}{9}$$

$$\cos\theta = \frac{-4}{9}$$

(b) [6 pts] A unit vector with the same direction as $\mathbf{a} = \langle -2, -1, 3\rangle$.

Let e_a be the unit vector in the ~~same~~ same direction as a.

$$e_a = \frac{1}{\|a\|} a = \frac{1}{\sqrt{(-2)^2+(-1)^2+3^2}} a = \frac{1}{\sqrt{14}} a = \frac{1}{\sqrt{14}} \langle -2, -1, 3\rangle$$

$$e_a = \frac{1}{\sqrt{14}} \langle -2, -1, 3\rangle$$

(c) [6 pts] The scalar component, $\text{comp}_{\mathbf{b}}\,\mathbf{a}$ of $\mathbf{a} = \langle -3, 1, 0\rangle$ along $\mathbf{b} = \langle 2, 2, 1\rangle$

$$\text{Comp}_b\, a = \frac{b \cdot a}{\|b\|} = \frac{(2)(-3)+(2)(1)+(1)(0)}{\sqrt{2^2+2^2+1^2}}$$

$$= \frac{-4}{\sqrt{9}} = \frac{-4}{3}$$

$$\text{comp}_b\, a = -\frac{4}{3}$$

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(d) [9 pts] The area of the triangle $\triangle PQR$ with vertices $P = (1,1,1)$, $Q = (1,-1,2)$ and $R = (3,1,3)$.

$\vec{V}_1 = \overrightarrow{PQ} = \langle 1-1, -1-1, 2-1 \rangle = \langle 0, -2, 1 \rangle$

$\vec{V}_2 = \overrightarrow{PR} = \langle 3-1, 1-1, 3-1 \rangle = \langle 2, 0, 2 \rangle$

Triangle Area $= \frac{1}{2}$ Parallelogram Area

$= \frac{1}{2} \left| \vec{V}_1 \times \vec{V}_2 \right|$

$= \frac{1}{2} \left| \left(\begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} \right) \right|$

$= \left| \frac{1}{2} (-4 + 2 - 2) \right| = \left| \frac{1}{2}(-4) \right| = |-2| = 2$

Triangle Area $= 2$ units^2

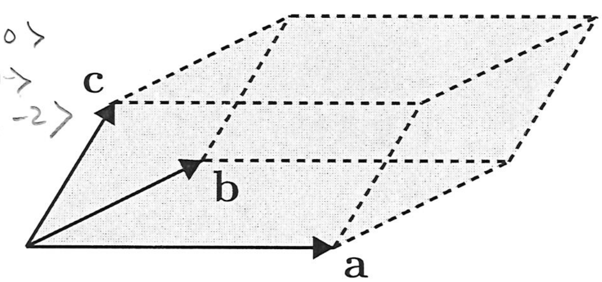
(e) [9 pts] The volume of the parallelepiped defined by the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (as shown).

$a = \langle 2, 1, 0 \rangle$

$b = \langle 0, 1, 1 \rangle$

$c = \langle 1, 4, -2 \rangle$

Volume $= | c \cdot (a \times b) |$



Volume $= \left| 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 4 \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \right|$

$= | 1(1) - 4(2) - 2(2) | = |-11| = 11$

Volume $= 11$ units^3

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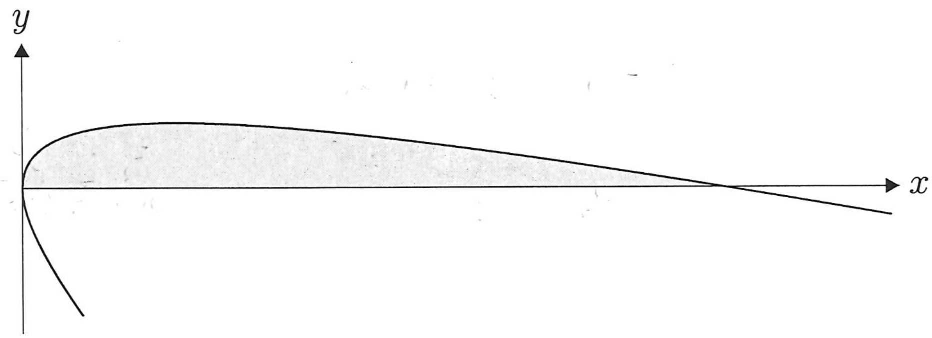
2. [20 pts] Consider the curve C in the xy-plane given by the parametric equations

$$x = 2t^3 + 6t^2, \qquad y = 2t - 2t^2.$$



(a) [8 pts] Find the slope of the tangent line to C at the point $(4, -4)$. (Hint: What value of t does this point correspond to?)

$4 = 2t^3 + 6t^2$ $\qquad$ $-4 = 2t - 2t^2$

$$\boxed{t = -1}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 - 4t}{6t^2 + 12t}$$

$$\frac{dy}{dx}(-1) = \frac{2 - 4(-1)}{6(-1)^2 + 12(-1)} = \frac{6}{-6} = -1$$

Slope $= -1$

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(b) [3 pts] Find the points where C intersects the x-axis. Which values of t correspond to these points?

$y = 2t - 2t^2$

$0 = 2t - 2t^2$

$0 = t(2-2t)$

$t = 0 \qquad 2-2t=0$

$-2t = -2$

$t = 1$

$t = \{0, 1\}$

For $t=0$

$x = 2(0)^3 + 6(0)^2 = 0$

$y = 2(0) - 2(0)^2 = 0$

$(0,0)$

For $t=1$

$x = 2(1)^3 + 6(1)^2 = 8$

$y = 2(1) - 2(1)^2 = 0$

$(8,0)$

Points:

$(0,0)$ for $t=0$

$(8,0)$ for $t=1$

(c) [9 pts] Find the area of the shaded region, bounded by C and the x-axis.

$$\text{Area} = \int_0^1 y(t)\,x'(t)\,dt = \int_0^1 (2t-2t^2)(6t^2+12t)\,dt$$

$$= \int_0^1 (12t^3 + 24t^2 - 12t^4 - 24t^3)\,dt$$

$$= \int_0^1 (-12t^4 - 12t^3 + 24t^2)\,dt$$

$$= -12\int_0^1 (t^4 + t^3 - 2t^2)\,dt = -12\left(\tfrac{1}{5}t^5 + \tfrac{1}{4}t^4 - \tfrac{2}{3}t^3\right)\Big|_0^1$$

$$= -12\left[\left(\tfrac{1}{5}(1)^5 + \tfrac{1}{4}(1)^4 - \tfrac{2}{3}(1)^3\right) - (0)\right]$$

$$= -12\left(\tfrac{1}{5} + \tfrac{1}{4} - \tfrac{2}{3}\right) = -12\left(\tfrac{12}{60} + \tfrac{15}{60} - \tfrac{40}{60}\right)$$

$$= -12\left(\tfrac{-13}{60}\right) = 12\left(\tfrac{13}{60}\right) = \tfrac{13}{5} \text{ units}^2$$

$$\text{Area} = \tfrac{13}{5} \text{ units}^2$$

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3. [20 pts] Consider the vector valued functions

$$\mathbf{r}_1(t) = \langle 2t+1, t^2-2t, t^2 \rangle, \qquad \mathbf{r}_2(t) = \langle 2t-1, t^2-3t+1, t-1 \rangle$$

3 -1 1

(a) [10 pts] The curves defined by these parametric equations intersect in exactly one point. Find the (x, y, z)-coordinates of this point.

$2t+1 = 2q-1$

$t^2 = q-1$

$q = t^2+1$

$2t+1 = 2(t^2+1)-1$

$2t+1 = 2t^2+2-1$

$0 = 2t^2-2t$

$0 = t(2t-2)$

$t = \{0, 1\}$

$t = 0$

$q = 0+1$

$q = 1$

$r(0) = \langle 1, 0, 0 \rangle$

If $t=1$, $q=2$

(b) [7 pts] Find tangent vectors to the two curves at the point of intersection you found in part (a).

$r_1'(t) = \langle 2, 2t-2, 2t \rangle$

$r_1'(0) = \langle 2, -2, 0 \rangle$

$r_2'(t) = \langle 2, 2t-3, 1 \rangle$

$r_2'(1) = \langle 2, -1, 1 \rangle$

(c) [3 pts] The *angle* between the two curves at an intersection point is defined to be the (acute) angle between the tangent lines at this point. Find the angle between the two curves at the intersection point you found in part (a).

$$\theta = \cos^{-1}\left(\frac{\langle 2,-2,0\rangle \cdot \langle 2,-1,1\rangle}{\|\langle 2,-2,0\rangle\| \, \|\langle 2,-1,1\rangle\|} \right)$$

$$\theta = \cos^{-1}\left(\frac{(2)(2)+(-2)(-1)+(0)(1)}{\sqrt{4+4}\ \sqrt{4+1+1}} \right)$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{8}\sqrt{6}}\right) = \cos^{-1}\left(\frac{6}{\sqrt{4}\sqrt{2}\sqrt{3}\sqrt{2}}\right) = \cos^{-1}\left(\frac{6}{4\sqrt{2}}\right) = \cos^{-1}\left(\frac{3}{2\sqrt{2}}\right)$$

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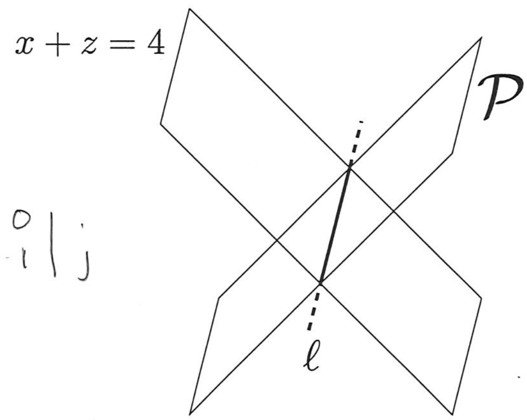
4. [15 pts] Let $\mathcal{P}$ be the plane which meets the plane $x + z = 4$ *at a right angle* along the line ℓ, defined by the vector valued function $\mathbf{r}(t) = \langle 1+t, 2+t, 3-t\rangle$. (That is, the two planes are perpendicular, and their intersection is the line ℓ.)

(a) [10 pts] Find a *normal* vector to $\mathcal{P}$.



$n = \langle 1, 0, 1\rangle \times \langle 1, 1, -1\rangle$

$n = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} j$

$n = -i + 2j + j$

$n = \langle -1, 2, 1\rangle$

(b) [5 pts] Find an equation (involving just the variables x, y and z, and no other variables) for $\mathcal{P}$.

$P = -1x + 2y + 1z = (-1)(1) + (2)(2) + (1)(3)$

$-x + 2y + z = 6$

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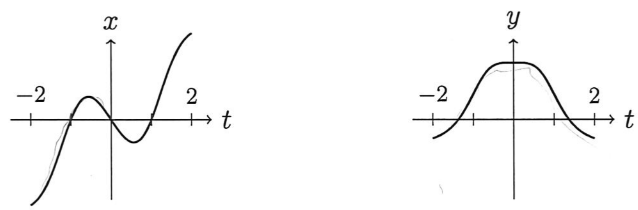
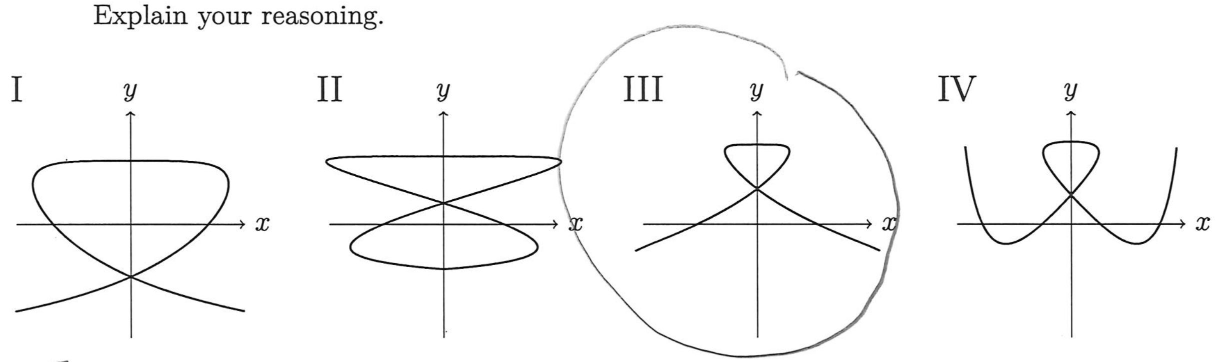
5. [10 pts] A curve C is defined by parametric equations in the form $x = x(t)$, $y = y(t)$, $-2 \le t \le 2$, where $x(t)$ and $y(t)$ have graphs:



One of the graphs labeled I-IV below is the graph of C in the xy-plane. Which one is it? Explain your reasoning.



III

X ~~go~~ cycles up and ~~down while~~ then down ~~while y~~ then up again, while y only goes up and then down.

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